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How Costly is it to Ignore Interest Rate Risk Management in your 401(k) Plan?

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Very preliminary and incomplete draft; please do not cite!

Abstract: This paper explicitly derives and explores optimal interest rate risk management for lifecycle investors in DC pension plans, and compares our results to the portfolio mix chosen in practice by Target-Date Fund (TDF) managers. We show that investments in long-term bonds play an important role in the portfolio of middle-aged individuals between ages 45 and 70. Our theoretical findings stand in sharp contrast with the investment choices made in practice; the role of long-term bonds is rather limited in the investment portfolios of 401(k) pension plan members in the US. Morningstar data on TDFs points out that the average bond duration is limited to five years and does not depend on age. We find that the absence of long-term bonds in the portfolio of a lifecycle investor can be costly, with the welfare loss peaking at 5 percent of consumption for middle-aged individuals.

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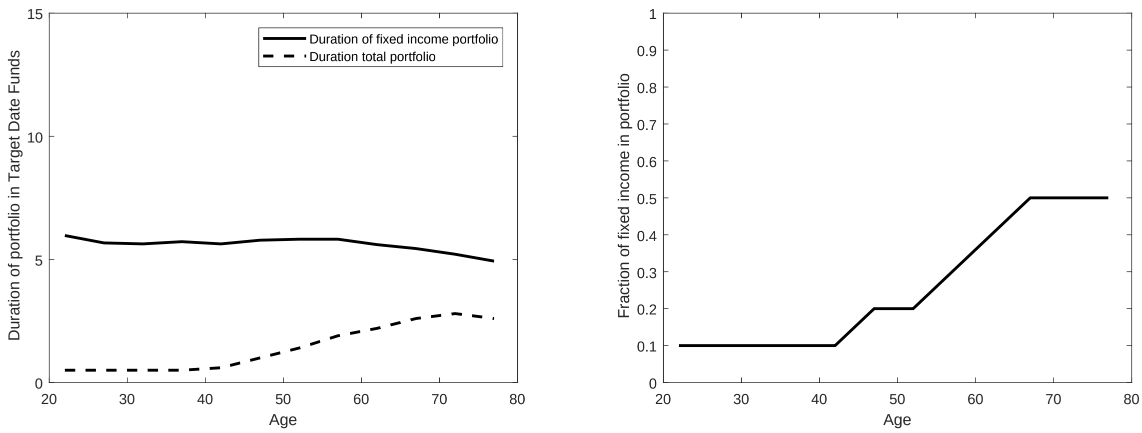
1 Introduction

Many occupational pension schemes around the world have shifted from guaranteed, defined benefit pension plans (DB) towards defined contribution pension plans (DC). The transition from DB to DC schemes is the result of a withdrawal of employers as risk sponsors. Many companies no longer wish to underwrite the risks of their pension funds, as these risks are too large in comparison to their core business. In DC pension contracts, all financial and demographic risks are borne by the plan participants. Participants save and invest in an individual account on the basis of traded financial assets. This development has increased the importance of an adequate design of lifecycle investment strategies, which prescribe a predetermined or dynamic rule for the portfolio choice of a plan participant as a function of age. In practice, Target Date Funds (TDFs) play an important role for participants in 401(k) DC pension plans in the US. These funds follow a predetermined reallocation of assets over the life cycle based on a specified retirement date. TDFs have become increasingly popular over the last two decades. In 2014, 72% of all 401(k) plans offered TDFs, 73% of 401(k) plan participants were offered TDFs, and 48% of 401(k) plan participants held assets in TDFs (Investment Company Institute (2016)). Furthermore, the share of 401(k) assets invested in TDFs grew from 5% in 2006 to 18% in 2014.

This paper explores optimal interest rate risk management for lifecycle investors in DC pension plans, and compares these results to the composition of the portfolio mix chosen in practice in TDFs. Our theoretical analysis derives the optimal portfolio allocation towards stocks, inflation-linked bonds, nominal bonds and cash for a lifecycle investor in the presence of stock market risk, interest rate risk and inflation risk. In a setting in which the investor is not subject to borrowing constraints, we arrive at analytical expressions for optimal interest rate management as a function of age. In a setting with borrowing constraints imposed on the investor, we provide numerical calculations.

We show that investments in long-term bonds, which provide protection against persistent shocks in interest rates, play a crucial role for middle-aged individuals in the age-period between 45 and 70. During this period of life, a substantial part of accumulated assets is invested in bonds and it is optimal to invest in long-term bonds with a long duration, in line with the duration of future consumption at those ages.

To compare our theoretical results to the investment choices of plan participants in practice, we turn to TDF data provided Morningstar which contains information on 2581 TDF funds from 61 TDF providers in the US. The data provides information on the portfolio share allocated to fixed income, the duration of fixed income, and the target date. By assuming that retirement takes place at the age of 65, the age of the investor can be calculated from the target date of the TDF. Figure 1 illustrates the fraction allocated to bonds (panel 1a) and the duration of those bonds (panel 1b) in Target Date Funds. The average bond duration is approximately five years, and does practically not depend on age. Hence, long-term bonds with a duration of say 20 or 30 years do not play a substantial role in most TDFs, also not at younger ages. Hence, the data points out that long-term bonds only play a limited role in TDFs in the US. Our theoretical finding on the importance of long-term bonds in the portfolio of middle-aged pension plan participants stands in sharp contrast with practice.



(a) Duration of fixed income portfolio in TDFs (b) Portfolio share of fixed income in TDFs

Figure 1. Duration of fixed income portfolio in TDFs. Panel (a) illustrates the duration of fixed income as a function of age for both the fixed income portfolio itself and the total portfolio (assuming the duration of other assets equal to zero). Panel (b) illustrates the portfolio share of fixed income as a function of age.

We calculate that absence of long-term bonds in the portfolio of a lifecycle investor can be costly, with the welfare loss peaking at 5 percent of consumption for middle-aged individuals. This welfare loss is derived from comparing the optimal investment strategy to a strategy in which we fix the duration of the fixed income portion of the portfolio to five (the median duration observed in the Morningstar data of TDFs).

Our paper contributes to the existing literature on optimal lifecycle investing, see, e.g., Bodie, Merton, and Samuelson (1992), Cocco, Gomes, and Maenhout (2005), and Gomes, Kotlikoff, and Viceira (2008). This literature has focussed primarily on studying the optimal lifecycle-pattern for stock market risk. We also contribute to the existing literature on optimal interest risk management, see, e.g., Viceira (2001), and Brennan and Xia (2002). We extend the analysis in the existing papers by introducing a lifecycle context. There have been a few earlier studies related to optimal interest rate risk management over the lifecycle, in particular Koijen, Nijman, and Werker (2010) and van Hemert (2010), but these papers do not provide analytical solutions.

2 Model

2.1 Preferences

Time is continuous. Denote by t adult age, which corresponds to effective age minus 20. For ease of exposition, we assume that the individual dies at adult age T. Let $c(t)$ and $\Pi(t)$ denote the individual's nominal consumption choice and the consumer price index at adult age t, respectively. The individual has CRRA preferences over real consumption. Hence, the individual's expected lifetime utility is given by

$$U = \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}\left(\frac{c(t)}{\Pi(t)}\right)^{1-\gamma}\mathrm{d}t\right], \tag{2.1}$$

where $\delta \geq 0$ denotes the subjective rate of time preference, γ corresponds to the coefficient of relative risk aversion, and $\mathbb{E}$ represents the (unconditional) expectation.

2.2 Asset Market and Wealth Accumulation

We consider a financial market with three state variables: the (instantaneous) real interest rate $r(t)$, the rate of inflation $\pi(t)$, and the nominal stock price $S(t)$. These state variables have the same dynamics as in Brennan and Xia (2002).[1] That is, the real interest rate and the rate of inflation follow Ornstein-Uhlenbeck processes and the nominal stock price evolves according to

[1] In contrast to Brennan and Xia (2002), we assume that the realized rate of inflation matches the expected rate of inflation. We can, however, extend our findings to the case where the realized rate of inflation does not necessarily coincide with the expected rate of inflation.

a geometric Brownian motion:

$$
\begin{aligned}
\mathrm{d}r(t) &= \kappa\left(\bar{r} - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r(t), && (2.2)\\
\mathrm{d}\pi(t) &= \theta\left(\bar{\pi} - \pi(t)\right)\mathrm{d}t + \sigma_\pi \mathrm{d}W_\pi(t), && (2.3)\\
\frac{\mathrm{d}S(t)}{S(t)} &= \left(R(t) + \lambda_S \sigma_S\right)\mathrm{d}t + \sigma_S \mathrm{d}W_S(t). && (2.4)
\end{aligned}
$$

Here, $\bar{r}$ and $\bar{\pi}$ denote long-term means, $\kappa \geq 0$ and $\theta \geq 0$ are mean reversion coefficients, $R(t) = r(t) + \pi(t)$ stands for the (instantaneous) nominal interest rate at adult age t, $\lambda_S \geq 0$ is the (constant) Sharpe ratio of the risky stock, $W(t) = (W_r(t), W_\pi(t), W_S(t))$ represents a vector of standard (possibly correlated) Brownian motions, and $\sigma = (\sigma_r, \sigma_\pi, \sigma_S) \geq 0$ is a vector of diffusion coefficients.[2] We summarize the (linear) correlation coefficients between the Brownian increments in the correlation matrix ρ:

$$
\rho = \begin{pmatrix} 1 & \rho_{r\pi} & \rho_{rS} \\ \rho_{r\pi} & 1 & \rho_{\pi S} \\ \rho_{rS} & \rho_{\pi S} & 1 \end{pmatrix}, \tag{2.5}
$$

where ρ_{ij} ($i, j \in \{r, \pi, S\}$, $i \neq j$) denotes the correlation coefficient between $\mathrm{d}W_i(t)$ and $\mathrm{d}W_j(t)$.

The nominal stochastic discount factor $m(t)$ satisfies (see, e.g., Brennan and Xia, 2002)

$$
\frac{\mathrm{d}m(t)}{m(t)} = -R(t)\mathrm{d}t + \phi^\top \mathrm{d}W(t). \tag{2.6}
$$

Here, $\top$ denotes the transpose sign and $\phi = (\phi_r, \phi_\pi, \phi_S)$ is a vector of factor loadings which determines the vector of market prices of risk associated with the underlying state variables. More specifically, we can obtain the vector of market prices of risk $\lambda = (\lambda_r, \lambda_\pi, \lambda_S)$ from the vector of factor loadings ϕ as follows:

$$
\lambda = -\rho\phi. \tag{2.7}
$$

Let $P_N(t, h)$ (respectively, $P_I(t, h)$) denote the price at adult age t of a nominal (respectively, an inflation-linked) zero-coupon bond maturing at adult age $t + h \geq t$. The individual invests

[2] For notational convenience, we often write a (column) vector in the form $z = (z_1, z_2, \ldots, z_n)$.

his total wealth – which equals the sum of human capital and financial wealth – in a nominal bond with fixed time to maturity h_N, an inflation-linked bond with fixed time to maturity h_I, a risky stock, and cash. The vector of risky asset prices $X(t) = \left(P_N\left(t, h_N\right), P_I\left(t, h_I\right), S(t)\right)$ satisfies the following dynamic equation (see the Appendix):

$$\frac{\mathrm{d}X(t)}{X(t)} = \mu(t)\mathrm{d}t + \Sigma \mathrm{d}W(t), \tag{2.8}$$

where

$$\mu(t) = \begin{pmatrix} R(t) - \lambda_r\sigma_r B\left(h_N\right) - \lambda_\pi\sigma_\pi C\left(h_N\right) \\ R(t) - \lambda_r\sigma_r B\left(h_I\right) \\ R(t) + \lambda_S\sigma_S \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} -B\left(h_N\right)\sigma_r & -C\left(h_N\right)\sigma_\pi & 0 \\ -B\left(h_I\right)\sigma_r & 0 & 0 \\ 0 & 0 & \sigma_S \end{pmatrix}.$$

Here, $B\left(h_N\right) = \left(1 - e^{-\kappa h_N}\right)/\kappa$, $B\left(h_I\right) = \left(1 - e^{-\kappa h_I}\right)/\kappa$, and $C\left(h_N\right) = \left(1 - e^{-\theta h_N}\right)/\theta$.

We denote by $\omega(t)$ and $A(t)$ the vector of portfolio weights and the individual's total wealth at adult age t, respectively. The individual's total wealth $A(t)$ satisfies the following dynamic budget constraint:

$$\mathrm{d}A(t) = \left(R(t) + \omega(t)^\top\left[\mu(t) - R(t)\right]\right)A(t)\mathrm{d}t + \omega(t)^\top\Sigma A(t)\mathrm{d}W(t) - c(t)\mathrm{d}t. \tag{2.9}$$

By integrating the increments $\mathrm{d}A(t)$, it follows that the individual's total wealth equals initial total wealth, plus any gains from trading, minus total consumption. The share of total wealth invested in cash at adult age t is given by $1 - \omega(t)^\top\mathbb{1}_3$ (here, $\mathbb{1}_3$ denotes a vector of length 3 consisting of ones).

2.3 Benchmark Parameter Values

In our numerical illustrations, we use the following benchmark parameter values:

Parameter	Value	Parameter	Value
$\bar{r} = r_0$	0.02	σ_S	0.2
κ	0.0347	ϕ_S	-0.2
σ_r	0.01	$\rho_{r\pi}$	0
ϕ_r	0.075	ρ_{rS}	0
$\bar{\pi} = \pi_0$	0.02	$\rho_{\pi S}$	0
θ	0.0693	γ	5
σ_π	0.01	δ	0.02
ϕ_π	0	T	65

Table 1. Benchmark parameter values. This table reports the parameter values that we use in our numerical illustrations.

2.4 Maximization Problem

The individual faces the following dynamic maximization problem:

$$\begin{aligned} &\underset{c(t),\omega(t):t\in[0,T]}{\text{maximize}} \quad \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}\left(\frac{c(t)}{\Pi(t)}\right)^{1-\gamma}\mathrm{d}t\right] \\ &\text{subject to} \quad \mathrm{d}A(t) = \big(R(t) + \omega(t)^\top\left[\mu(t) - R(t)\right]\big)\,A(t)\mathrm{d}t + \omega(t)^\top\Sigma A(t)\mathrm{d}W(t) - c(t)\mathrm{d}t. \end{aligned} \tag{2.10}$$

Section 3 analyzes and discusses the optimal policies over the life cycle. Section 5 explores the impact of a borrowing constraint on the optimal life cycle patterns.

3 Optimal Life Cycle Policies

3.1 Optimal Consumption Strategy

We determine the optimal (nominal) consumption choice $c^*(t)$ using the martingale approach (Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989), and Cox and Huang (1991)). After we have solved for the optimal consumption choice $c^*(t)$, we determine the vector of optimal portfolio weights $\omega^*(t)$ using replication arguments. We find that the individual's optimal consumption choice is given by (see the Appendix for a proof)

$$c^*(t) = c^*(0)\exp\left\{\int_0^t \pi(s)\,\mathrm{d}s + \frac{1}{\gamma}\int_0^t\left(r(s) + \frac{1}{2}\phi^\top\rho\phi - \delta\right)\mathrm{d}s - \frac{1}{\gamma}\phi^\top W(t)\right\}. \tag{3.1}$$

Here, $c^*(0)$ denotes the individual's consumption choice at the beginning of the life cycle which is determined such that the market-consistent value of the optimal consumption stream equals the individual's initial total wealth:

$$\mathbb{E}\left[\int_0^T m(t)c^*(t)\mathrm{d}t\right] = A(0). \tag{3.2}$$

The change in log consumption consists of three parts (this follows from (3.1)):

$$\mathrm{d}\log c^*(t) = \pi(t)\,\mathrm{d}t + \frac{1}{\gamma}\left(r(t) + \frac{1}{2}\phi^\top\rho\phi - \delta\right)\mathrm{d}t - \frac{1}{\gamma}\phi^\top\,\mathrm{d}W(t). \tag{3.3}$$

The first part of (3.3) reflects the individual's desire to maintain the purchasing power of pre- and post-retirement consumption. The second part models the expected growth rate of log *real* consumption. Whether this part is positive or negative depends upon the relative strength of two counteracting forces. On the one hand, the individual is impatient: he has a preference to consume sooner rather than later. This has a negative effect on the expected growth rate of log real consumption. On the other hand, by giving up consumption today, the individual will be able to consume more in the future. This has a positive effect on the expected growth rate of log real consumption. The last part denotes the impact of unexpected shocks in the underlying state variables on log consumption. By allowing unexpected shocks in the underlying state variables to affect log consumption, the individual can achieve a higher expected consumption stream. Finally, we note that the preference parameter γ plays two distinct roles in the optimal consumption rule (3.3). On the one hand, it models the degree of relative risk aversion; that is, the exposure of log consumption to unexpected shocks in the underlying state variables. On the other hand, it models the elasticity of intertemporal substitution; that is, the extent to which the expected growth rate of log real consumption changes following a change in the real interest rate.

3.2 Optimal Portfolio Strategy

We determine the vector of optimal portfolio weights $\omega^*(t)$ such that changes in the market-consistent value of optimal remaining lifetime consumption exactly matches changes in the value of the individual's investment portfolio. The vector of optimal portfolio weights $\omega^*(t)$ is given by (see the Appendix for a proof)

$$\omega_N^*(t) = \frac{1}{\gamma} \frac{\phi_\pi}{C(h_N)\,\sigma_\pi}, \tag{3.4}$$

$$\omega_I^*(t) = \frac{1}{B(h_I)} \left[\frac{1}{\gamma} \frac{\phi_r}{\sigma_r} - \omega_N^*(t) B(h_N) \right] + \frac{D_{K^*}(t)}{B(h_I)}, \tag{3.5}$$

$$\omega_S^*(t) = -\frac{1}{\gamma} \frac{\phi_S}{\sigma_S}. \tag{3.6}$$

Here, $D_{K^*}(t)$ denotes the duration (i.e., real interest rate sensitivity) of the optimal conversion factor $K^*(t) = A^*(t)/c^*(t)$ (where $A^*(t)$ represents the total wealth level implied by implementing the optimal consumption and portfolio policies). The Appendix provides a mathematical definition of $D_{K^*}(t)$ (see (A16)).

The first element of $\omega^*(t)$, i.e., $\omega_N^*(t)$, denotes the share of total wealth invested in a nominal bond with fixed time to maturity h_N. The individual allocates part of his total wealth to a nominal bond for a speculative reason: he wants to benefit from the inflation risk premium $-\lambda_\pi \sigma_\pi C(h_N) \geq 0$. The size of this speculative demand is positively related to the unit factor loading ϕ_π/σ_π and negatively related to the individual's coefficient of relative risk aversion γ. Because these parameters are assumed constant and the time to maturity h_N is fixed, the share of total wealth invested in the nominal bond does not change over the individual's life cycle.

The second element of $\omega^*(t)$, i.e., $\omega_I^*(t)$, denotes the share of total wealth invested in an inflation-linked bond with fixed time to maturity h_I. There are two reasons why the individual prefers to allocate part of his total wealth to an inflation-linked bond. The first reason is a speculative one: the individual wants to profit from the real interest rate risk premium $-\lambda_r \sigma_r B(h_I) \geq 0$. However, because the individual already picks up the real interest rate risk premium $-\lambda_r \sigma_r B(h_N) \geq 0$ by investing part of his total wealth in a nominal bond with fixed time to maturity h_N, the speculative demand for the inflation-linked bond can become negative for sufficiently large ϕ_π/σ_π or sufficiently small ϕ_r/σ_r (see also (3.4) and (3.5)).

The second reason to invest in an inflation-linked bond is a hedging one: the individual wants to hedge against real interest rate fluctuations in the conversion factor. This so-called intertemporal hedging demand depends on the duration of the conversion factor $D_{K^*}(t)$. We note that the duration of optimal total wealth $D_{A^*}(t) = D_{K^*}(t) + \phi_r/(\gamma\sigma_r)$ is (typically) larger than the duration of the conversion factor $D_{K^*}(t)$, because the former also takes into account the impact of a change in the real interest rate on current consumption.

Figure 2 shows the (median) duration of the conversion factor over the life cycle for various levels of relative risk aversion. As shown by this figure, the duration declines as the individual becomes older. Indeed, the younger the individual, the longer his investment period, and hence the more sensitive the conversion factor is to changes in the real interest rate. The duration also depends on the level of relative risk aversion. In particular, the duration of a moderately risk averse individual is smaller than the duration of a highly risk averse individual. Intuitively, after a persistent increase in the real interest rate, an individual with a moderate level of risk aversion increases the expected growth rate of real consumption more than an individual with a high level of risk aversion. As a result, the conversion factor of a moderately risk averse individual responds less to a change in the real interest rate than the conversion factor of a highly risk averse individual. In the extreme case in which the individual is infinitely risk averse, the duration of the conversion factor coincides with the duration of an inflation-linked perpetuity.

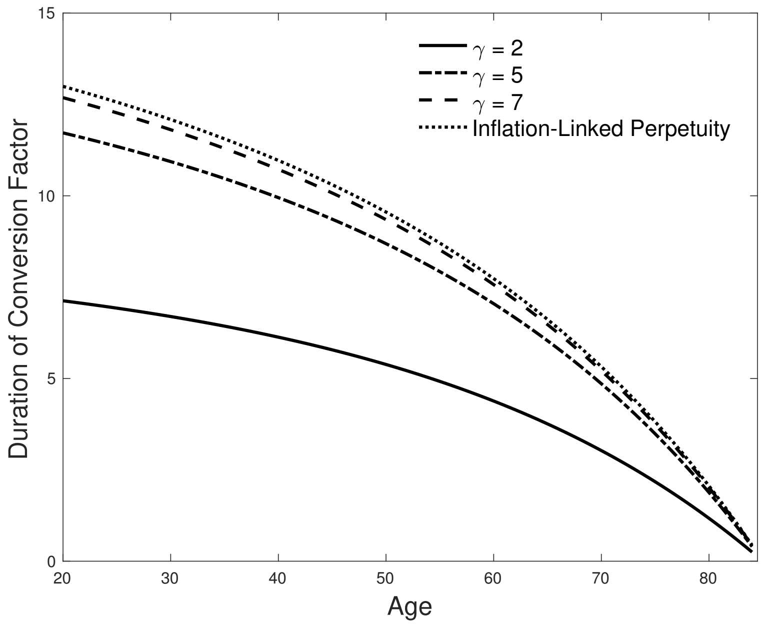


Figure 2. Illustration of the duration of the conversion factor. The figure illustrates the median duration of the conversion factor over the life cycle for various levels of relative risk aversion. The benchmark parameter values are given in Table 1.

Finally, the third element of $\omega^*(t)$, i.e., $\omega_S^*(t)$, denotes the share of total wealth invested in the risky stock. The individual invests part of his total wealth in the risky stock so as to pick up the equity risk premium $\lambda_S \sigma_S \geq 0$. As is also the case for the (speculative) nominal bond demand, the share of total wealth invested in the risky stock does not change over the individual's life cycle.

Figure 3 illustrates the median portfolio shares over the life cycle. The individual invests his total wealth in a 10-year nominal bond, a 30-year inflation-linked bond, a risky stock, and cash. As shown by this figure, the demand for the inflation-linked bond decreases as the individual becomes older. An inflation-linked bond – which hedges against real interest rate risk as well as inflation risk – is particularly valuable for a young individual because the market-consistent value of his optimal remaining lifetime consumption is very sensitive to changes in the real interest rate and the inflation rate. Finally, we note that the demand for the nominal bond is zero because the unit factor loading $\phi_\pi/\sigma_\pi = 0$. The next section explores how the individual should allocate his financial wealth over the various asset classes.

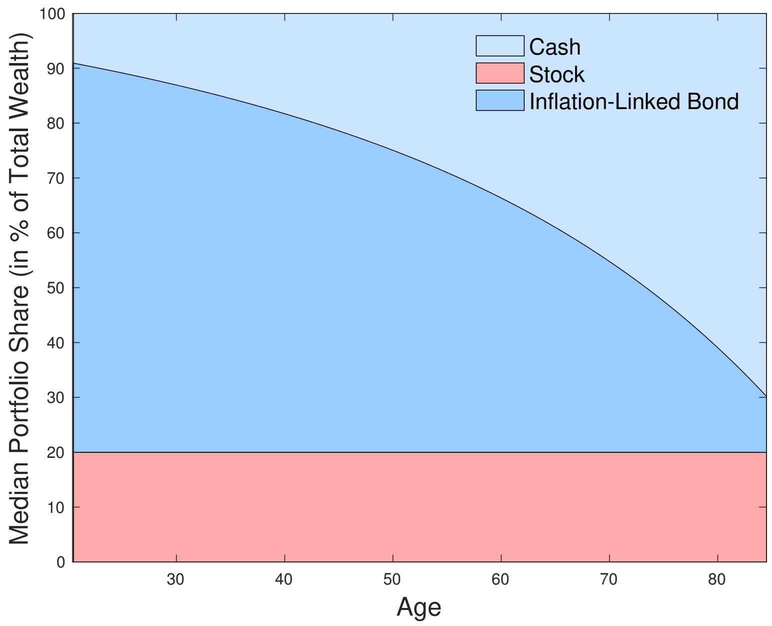


Figure 3. Illustration of median portfolio shares over the life cycle. The figure illustrates the median shares of total wealth invested in a 10-year nominal zero-coupon bond, a 30-year inflation-linked zero-coupon bond, a risky stock and cash as a function of age. The benchmark parameter values are given in Table 1. We note that the demand for the nominal bond is zero because the unit factor loading $\phi_\pi/\sigma_\pi = 0$.

4 Impact of Human Capital on the Optimal Portfolio Strategy

This section explores the impact of human capital (i.e., discounted value of future labor earnings) on the optimal portfolio strategy. Inspired by Bodie et al. (1992), we assume that labor income is risk-free in real terms. Furthermore, we allow pension payments from the government to be included in the computation of human capital. The state pension payments are also assumed to be risk-free in real terms. We can thus interpret human capital as an inflation-linked bond. Let $D_H(t)$ denote the duration at time t of human capital. The Appendix provides a mathematical definition of $D_H(t)$ (see (A25)). The individual can only invest his financial wealth – which equals total wealth minus human capital – in the financial market. When the individual makes the asset allocation decision, he takes into account the fact that he already possess a valuable asset, namely human capital. Let $\widehat{\omega}(t) = (\widehat{\omega}_N(t), \widehat{\omega}_I(t), \widehat{\omega}_S(t))$ be the vector consisting of the shares of *financial* wealth invested in the risky assets. We find that the vector of optimal portfolio weights $\widehat{\omega}^*(t)$ is given by (see the Appendix for a proof)

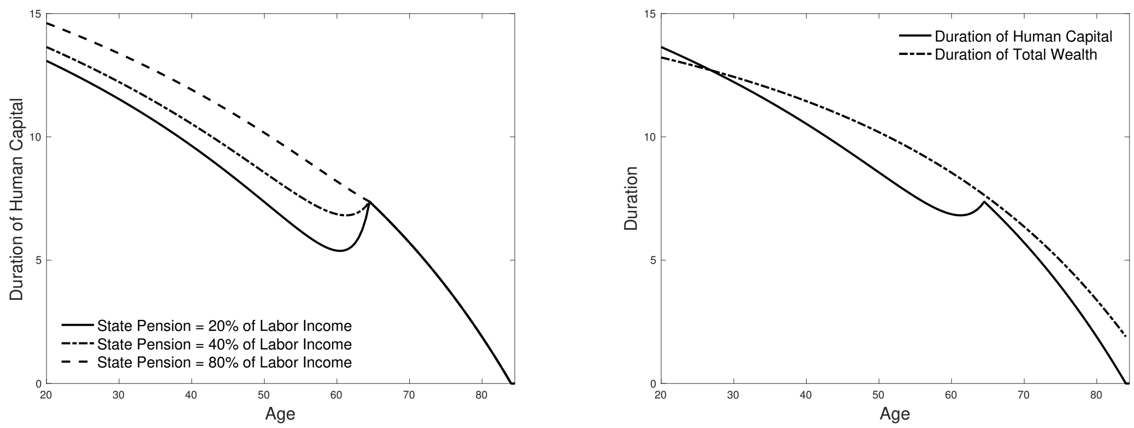
$$\widehat{\omega}^*(t) = \omega^*(t) + \frac{H(t)}{F(t)}\omega^*(t) - \left(0, \frac{D_H(t)}{B(h_I)}\frac{H(t)}{F(t)}, 0\right), \tag{4.1}$$

where $H(t)$ and $F(t)$ denote human capital and financial wealth at adult age t, respectively.

The shares of financial wealth invested in the nominal bond and the risky stock are not constant, but rather decrease (on average) with age. Intuitively, because human capital does not carry inflation risk and stock market risk, its expected return is relatively low. As a result, to obtain a sufficiently high expected return on total wealth, financial wealth should be tilted towards investments in the nominal bond and the risky stock. Because human capital becomes relatively less important as the individual ages, the shares of financial wealth invested in the nominal bond and the risky stock decrease over the individual's life cycle. The impact of human capital on the demand for the inflation-linked bond is less clear. We can write the share of financial wealth invested in the inflation-linked bond as follows:

$$\widehat{\omega}_I^*(t) = \omega_I^*(t) + \frac{H(t)}{F(t)B(h_I)}\left(D_{A^*}(t) - \omega_N^*(t)B(h_N) - D_H(t)\right). \tag{4.2}$$

Human capital is already exposed to real interest rate risk. However, its duration $D_H(t)$ is typically not equal to the difference between the duration of total wealth $D_{A^*}(t)$ and the duration of the nominal bond portfolio $\omega_N^*(t)B(h_N)$. If $D_H(t)$ is smaller than $D_{A^*}(t) - \omega_N^*(t)B(h_N)$, then human capital has 'insufficient' exposure to real interest rate risk. As a result, the individual should increase the investments in the inflation-linked bond to obtain an adequate exposure to real interest rate risk. Conversely, if $D_H(t)$ is larger than $D_{A^*}(t) - \omega_N^*(t)B(h_N)$, then human capital has too much exposure to real interest rate risk. In that case, the individual should reduce the investments in the inflation-linked bond to achieve the desired exposure to real interest rate risk. Finally, we note that the duration of human capital $D_H(t)$ is not always smaller than the duration of total wealth $D_{A^*}(t)$. In particular, the duration of human capital typically exceeds the duration of total wealth if the size of the state pension is sufficiently large and the coefficient of relative risk aversion is relatively low. Figure 4 shows the (median) duration of human capital over the life cycle for various levels of state pension. This figure also compares the duration of human capital with the duration of total wealth.

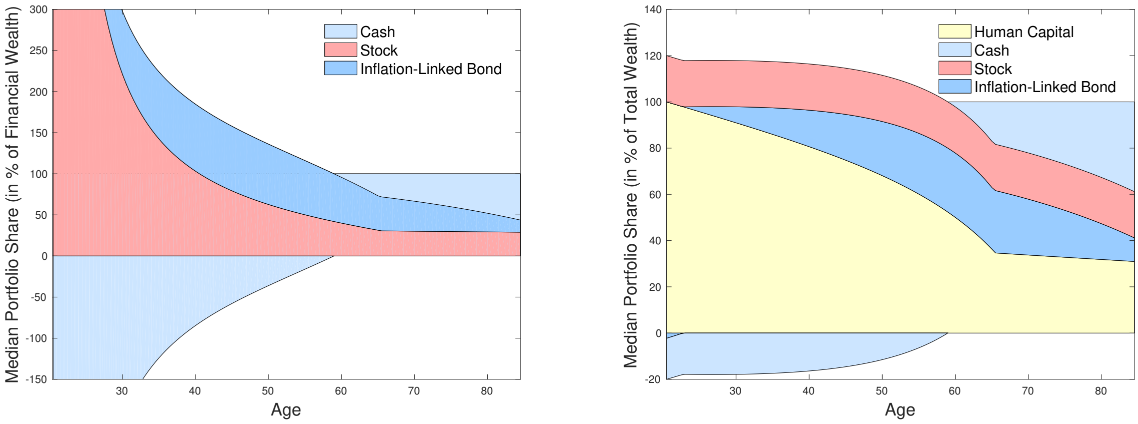


(a)

(b)

Figure 4. Illustration of the duration of human capital. Panel (a) illustrates the (median) duration of human capital over the life cycle for various levels of state pension. Panel (b) compares the (median) duration of human capital with the (median) duration of total wealth. This panel assumes that the state pension is equal to 40% of labor income. The benchmark parameter values are given in Table 1. We note that the demand for the nominal bond is zero because the unit factor loading $\phi_\pi/\sigma_\pi = 0$.

Figure 5(a) illustrates the median shares of financial wealth invested in a 10-year nominal bond, a 30-year inflation-linked bond, a risky stock, and cash. As shown by this figure, the investment portfolios of young individuals consist not only of risky stocks but also of (index-linked) bonds. This stands in sharp contrast with empirical evidence: bond portfolios are typically absent in the investment portfolios of young individuals. The demand for cash is negative for most ages: the individual borrows money to invest in the financial market. Figure 5(b) shows how total wealth is allocated between human capital, the nominal bond portfolio, the inflation-linked bond portfolio, the stock portfolio, and the cash account.



(a) Composition of Financial Wealth

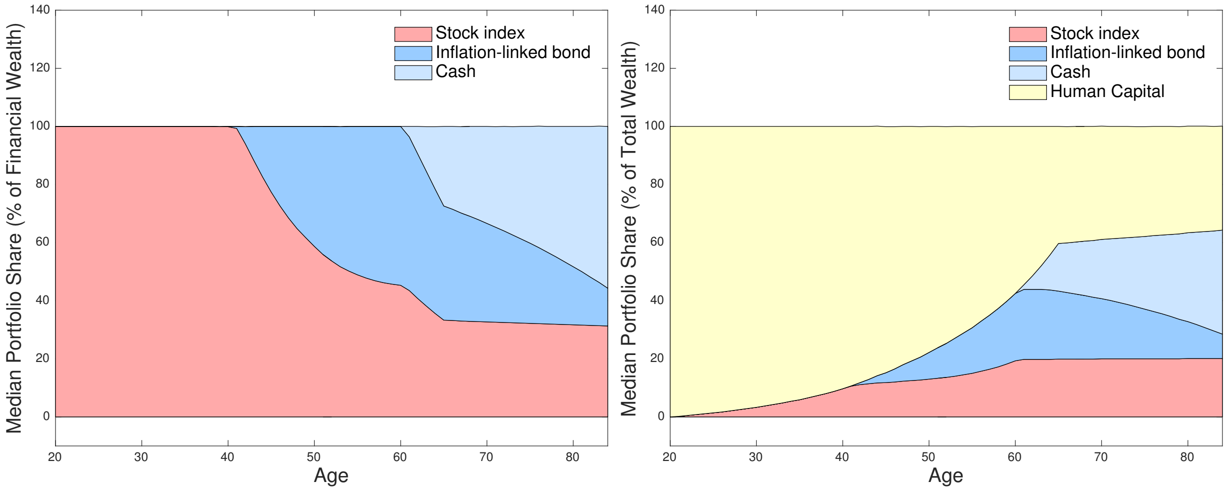
(b) Composition of Total Wealth

Figure 5. Illustration of median portfolio shares over the life cycle. Panel (a) illustrates the median shares of financial wealth invested in a 10-year nominal zero-coupon bond, a 30-year inflation-linked zero-coupon bond, a risky stock and cash as a function of age. Panel (b) illustrates the composition of total wealth. The benchmark parameter values are given in Table 1. The state pension is assumed to be 40% of labor income. We note that the demand for the nominal bond is zero because the unit factor loading $\phi_\pi/\sigma_\pi = 0$.

5 Borrowing Constraint

In section 3 we saw that, once we add human capital to the model, the financial portfolio becomes highly leveraged whenever human capital form a big part of total wealth. One may worry that such levels of financial leverage are hard to obtain in practice, as human capital can not be used as collateral. Therefore, we now explore the impact of a borrowing constraint and short-sales constraint on the optimal portfolio policies. The constraints imply that we can no longer derive the optimal policies in closed-form. Hence, we resort to numerical optimization. The Appendix outlines the numerical solution technique.

Figure 6(a) illustrates the median shares of financial wealth invested in each asset. In a



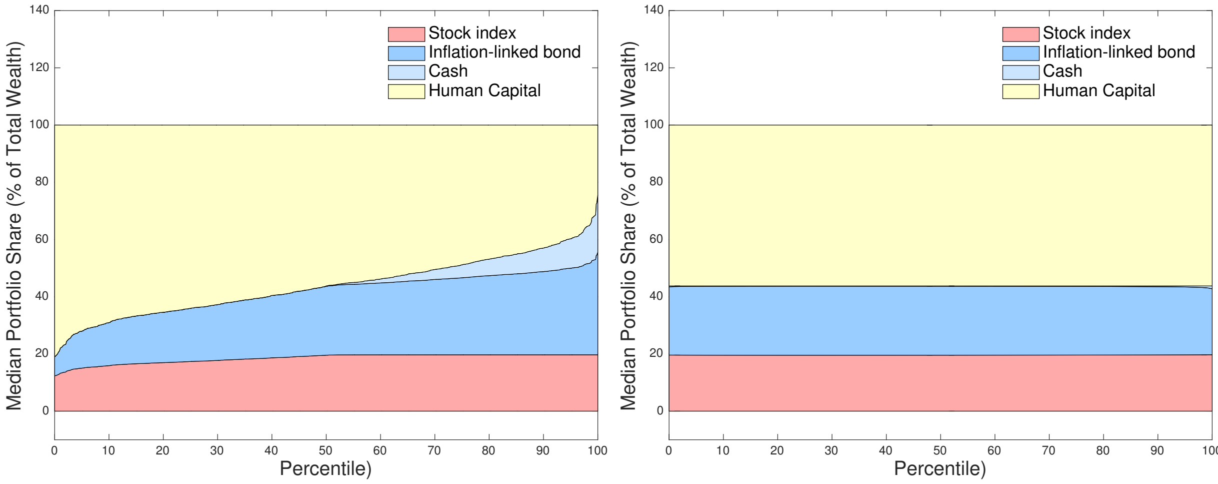
(a) Composition of Financial Wealth (b) Composition of Total Wealth

Figure 6. Illustration of constrained median portfolio shares over the life cycle. Panel (a) illustrates the median shares of financial wealth invested in a 10-year nominal zero-coupon bond, a 30-year inflation-linked zero-coupon bond, the stock index and cash, as a function of age. Panel (b) illustrates the composition of total wealth. The benchmark parameter values are given in Table 1. The state pension is assumed to be 40% of labor income. We note that the demand for the nominal bond is zero because the unit factor loading $\phi_\pi/\sigma_\pi = 0$.

median scenario, the portfolio constraints are binding all the way up to an age of 60. Above this age, the solution is equal to the unconstrained solution as described in Section 3. Below this age, the portfolio can be roughly described by following a rule of thumb: scarce portfolio space is first allocated to stocks, up to the unconstrained optimal level, and then to the inflation-linked bond. This is only approximately true though, at the age of sixty it is clearly visible that the optimal allocation to stocks kinks, which shows that the constraint also reduces the stock exposure at this point.

Notice that the age at which the constraint is binding depends on the realization of returns over the life-cycle. To illustrate this, Figure 7(a) shows the composition of total wealth at age 60 as a function of realized wealth. The horizontal axis denotes the percentiles of the wealth distribution. Above the median, at this age, the constraint is no longer binding.

Strictly speaking, the optimal portfolio allocation also depends on the realized level of the real interest rate. The interest rate affects the duration of optimal consumption and hence the hedging demand for the long-term bond. Figure 7(b) however illustrates that this impact is hardly observable. The reason is that the change in duration is largely offset by the fact that the duration of human capital also changes.



(a) as function of realized wealth

(b) as function of realized interest rate

Figure 7. Illustration of constrained median portfolio shares at age 60. Panel (a) illustrates the composition of total wealth at age 60 as a function of the distribution of wealth (conditional on the real interest rate at its median). Panel (b) illustrates the composition of total wealth at age 60 as a function of the real interest rate (with the financial wealth share fixed at its unconditional median). The distributions are based on Monte-Carlo simulation with 50,000 realized scenarios.

5.1 Welfare Cost of Sub-Optimal Interest Hedge

Not appropriately hedging interest rate risk can be a costly mistake within the model. To illustrate this, we calculate what the welfare effect is if the duration of the fixed income portfolio (long-term bond and cash), is not chosen optimally for the next year. Instead of the (constrained) optimal fixed income portfolio, we assume that the duration of the fixed income portfolio is fixed at five (as observed in the Morningstar data on TDFs). Figure 8 shows the welfare effect over the life-cycle (at median wealth and interest). The welfare effect is expressed as a percentage of consumption in the corresponding year. Due to the portfolio constraints, young individuals do not invest in fixed income to begin with, hence the welfare effect is zero at young ages. Especially for middle aged individuals, the mistake can be costly, with the welfare loss peaking at five percent of consumption for our baseline parameters. The figure also shows how this value varies if we consider interest rate shocks to be more or less persistent, by changing the halftime of interest rate shocks to either 5 or 35. Since there is no reason to believe that interest rates are actually mean-reverting in practice, probably a high interest rate persistence better reflects the fundamental uncertainty investors face about long-term interest rate developments.

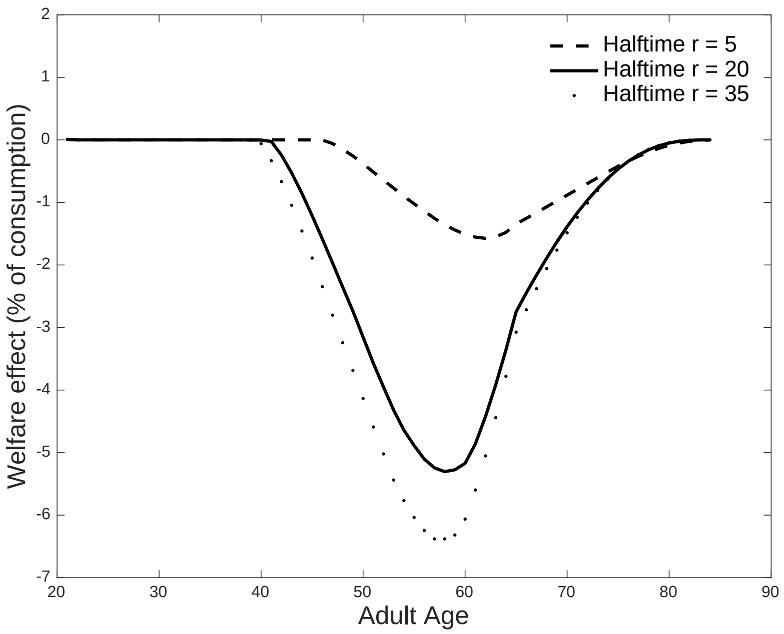


Figure 8. Welfare effect from static fixed income portfolio The figure illustrates the welfare loss from not implementing the (constrained) optimal fixed income portfolio, but investing in a fixed income portfolio with a fixed duration of five instead during the next year (and investing optimally again after that). The welfare loss is expressed as a percentage of consumption in the affected year.

6 Conclusion

PM

Appendix A: Mathematical Proofs

Derivation of (2.8)

This appendix derives the dynamics of the price of a nominal bond. We can obtain the dynamics of the price of an inflation-linked bond using similar techniques. The bond price $P_N(t,h)$ follows from computing the following conditional expectation:[3]

$$\begin{aligned} P_N(t,h) &= \mathbb{E}_t\left[\frac{m(t+h)}{m(t)}\right] \\ &= \mathbb{E}_t\left[\exp\left\{-\int_0^h \left(r(t+v)+\pi(t+v)+\frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}v+\phi^\top\int_0^h \mathrm{d}W(t+v)\right\}\right]. \end{aligned} \tag{A1}$$

Here, $\mathbb{E}_t$ denotes the expectation conditional upon the information available at time t.

Equation (A1) shows that the aggregate real interest rate $\bar{r}(t,h)=\int_0^h r(t+v)\mathrm{d}v$ and the aggregate inflation rate $\bar{\pi}(t,h)=\int_0^h \pi(t+v)\mathrm{d}v$ play a key role in determining the nominal bond price. We find that the aggregate real interest rate $\bar{r}(t,h)$ is given by

$$\begin{aligned} \bar{r}(t,h) &= \int_0^h r(t+v)\mathrm{d}v \\ &= \int_0^h \left(e^{-\kappa v}r(t)+\left(1-e^{-\kappa v}\right)\bar{r}\right)\mathrm{d}v+\sigma_r\int_0^h\int_0^v e^{-\kappa(v-u)}\mathrm{d}W_r(t+u)\mathrm{d}v \\ &= \int_0^h \left(r(t)+\left(1-e^{-\kappa v}\right)\left(\bar{r}-r(t)\right)\right)\mathrm{d}v+\sigma_r\int_0^h\int_v^h e^{-\kappa(h-u)}\mathrm{d}u\mathrm{d}W_r(t+v) \\ &= \int_0^h \left(r(t)+\kappa B(v)\left(\bar{r}-r(t)\right)\right)\mathrm{d}v+\sigma_r\int_0^h \frac{1}{\kappa}\left(1-e^{-\kappa(h-v)}\right)\mathrm{d}W_r(t+v) \\ &= \int_0^h \mathbb{E}_t\left[r(t+v)\right]\mathrm{d}v+\sigma_r\int_0^h B(h-v)\mathrm{d}W_r(t+v). \end{aligned} \tag{A2}$$

The second equality in (A2) follows from the fact that

$$\begin{aligned} r(t+v) &= e^{-\kappa v}r(t)+\left(1-e^{-\kappa v}\right)\bar{r}+\sigma_r\int_0^v e^{-\kappa(v-u)}\mathrm{d}W_r(t+u) \\ &= \mathbb{E}_t\left[r(t+v)\right]+\sigma_r\int_0^v e^{-\kappa(v-u)}\mathrm{d}W_r(t+u). \end{aligned} \tag{A3}$$

We can derive (A3) by repeated substitution. In a similar fashion, we find that the aggregate

[3] The bond price $P_I(t,h)$ follows from computing the conditional expectation $\mathbb{E}_t\left[\frac{m(t+h)}{m(t)}\Pi(t+h)\right]$.

inflation rate $\bar{\pi}(t,h)$ is given by

$$\bar{\pi}(t,h) = \int_0^h \pi(t+v)\mathrm{d}v = \int_0^h \mathbb{E}_t\left[\pi(t+v)\right]\mathrm{d}v + \sigma_\pi \int_0^h C(h-v)\mathrm{d}W_\pi(t+v). \tag{A4}$$

Substituting (A2) and (A4) into (A1) to eliminate $\int_0^h r(t+v)\mathrm{d}v$ and $\int_0^h \pi(t+v)\mathrm{d}v$, we arrive at

$$\begin{aligned}
P_N(t,h) &= \exp\left\{-\int_0^h \left(\mathbb{E}_t\left[r(t+v)+\pi(t+v)\right] + \frac{1}{2}\phi^\top \rho \phi\right)\mathrm{d}v\right\} \\
&\quad \mathbb{E}_t\left[\exp\left\{\int_0^h \phi_S \mathrm{d}W_S(t+v) + \int_0^h \left(\phi_r - B(h-v)\sigma_r\right)\mathrm{d}W_r(t+v)\right.\right. \\
&\qquad \left.\left. + \int_0^h \left(\phi_\pi - C(h-v)\sigma_\pi\right)\mathrm{d}W_\pi(t+v)\right\}\right] \\
&= \exp\left\{-\int_0^h \left(\mathbb{E}_t\left[R(t+v)\right] - \lambda_r\sigma_r B(v) - \lambda_\pi\sigma_\pi C(v)\right.\right. \\
&\qquad \left.\left. -\frac{1}{2}B^2(v)\sigma_r^2 - \frac{1}{2}C^2(v)\sigma_\pi^2 - \rho_{r\pi}B(v)C(v)\sigma_r\sigma_\pi\right)\mathrm{d}v\right\} \\
&= \exp\left\{-\int_0^h R(t,v)\mathrm{d}v\right\}.
\end{aligned} \tag{A5}$$

Here, the instantaneous nominal forward interest rate at adult age t for horizon v, i.e., $R_{t,v}$, is defined as follows:

$$R(t,v) = \mathbb{E}_t\left[R(t+v)\right] - \lambda_r\sigma_r B(v) - \lambda_\pi\sigma_\pi C(v) - \frac{1}{2}B^2(v)\sigma_r^2 - \frac{1}{2}C^2(v)\sigma_\pi^2 - \rho_{r\pi}B(v)C(v)\sigma_r\sigma_\pi. \tag{A6}$$

The log bond price is given by (this follows from (A5) and (A6))

$$\begin{aligned}
\log P_N(t,h) = -\int_0^h &\Big(r(t) + \kappa B(v)\left(\bar{r} - r(t)\right) + \pi(t) + \theta C(v)\left(\bar{\pi} - \pi(t)\right) - \lambda_r\sigma_r B(v) \\
&- \lambda_\pi\sigma_\pi C(v) - \frac{1}{2}B^2(v)\sigma_r^2 - \frac{1}{2}C^2(v)\sigma_\pi^2 - \rho_{r\pi}B(v)C(v)\sigma_r\sigma_\pi\Big)\mathrm{d}v.
\end{aligned} \tag{A7}$$

Solving the integral (A7), we arrive at[4]

$$
\begin{aligned}
\log P_N(t,h) &= -r(t)h - (\bar{r} - r(t))\,(h - B(h)) - \pi(t)h - (\bar{\pi} - \pi(t))\,(h - C(h)) \\
&\quad + \frac{\lambda_r \sigma_r}{\kappa}\,(h - B(h)) + \frac{\lambda_\pi \sigma_\pi}{\theta}\,(h - C(h)) \\
&\quad + \frac{1}{2}\frac{\sigma_r^2}{\kappa^2}\left(h - 2B(h) + \frac{1}{2}B(2h)\right) + \frac{1}{2}\frac{\sigma_\pi^2}{\theta^2}\left(h - 2C(h) + \frac{1}{2}C(2h)\right) \\
&\quad + \frac{\rho_{r\pi}\sigma_r\sigma_\pi}{\kappa\theta}\left(h - B(h) - C(h) + \frac{1}{\kappa+\theta}\left(1 - e^{-(\kappa+\theta)h}\right)\right) \\
&= -r(t)B(h) - \pi(t)C(h) - M(h).
\end{aligned}
\tag{A8}
$$

Here, $M(h)$ is defined as follows:

$$
\begin{aligned}
M(h) &= \left(\bar{r} - \frac{\lambda_r \sigma_r}{\kappa} - \frac{1}{2}\frac{\sigma_r^2}{\kappa^2}\right)(h - B(h)) + \frac{1}{4\kappa}B^2(h)\sigma_r^2 \\
&\quad + \left(\bar{\pi} - \frac{\lambda_\pi \sigma_\pi}{\theta} - \frac{1}{2}\frac{\sigma_\pi^2}{\theta^2}\right)(h - C(h)) + \frac{1}{4\theta}C^2(h)\sigma_\pi^2 \\
&\quad + \frac{\rho_{r\pi}\sigma_r\sigma_\pi}{\kappa\theta}\left(h - B(h) - C(h) + \frac{1}{\kappa+\theta}\left(1 - e^{-(\kappa+\theta)h}\right)\right).
\end{aligned}
\tag{A9}
$$

To calculate how the price of a nominal bond with a fixed maturity date $t+h$ develops as time proceeds (i.e., $t+h$ is fixed but t changes), we apply Itô's lemma to

$$
P_N(t,h) = \exp\left\{-r(t)B(h) - \pi(t)C(h) - M(h)\right\}. \tag{A10}
$$

We find

$$
\begin{aligned}
\frac{\mathrm{d}P_N(t,h)}{P_N(t,h)} &= \Big(R(t,h) - \kappa B(h)\,(\bar{r} - r(t)) - \theta C(h)\,(\bar{\pi} - \pi(t)) \\
&\qquad + \frac{1}{2}B^2(v)\sigma_r^2 + \frac{1}{2}C^2(v)\sigma_\pi^2 + \rho_{r\pi}B(v)C(v)\sigma_r\sigma_\pi\Big)\mathrm{d}t \\
&\quad - B(h)\sigma_r \mathrm{d}W_r(t) - C(h)\sigma_\pi \mathrm{d}W_\pi(t) \\
&= (R(t) - \lambda_r\sigma_r B(h) - \lambda_\pi\sigma_\pi C(h))\,\mathrm{d}t - B(h)\sigma_r \mathrm{d}W_r(t) - C(h)\sigma_\pi \mathrm{d}W_\pi(t).
\end{aligned}
\tag{A11}
$$

[4]The first equality follows from $B^2(v) = \left(1 - 2e^{-\kappa v} + e^{-2\kappa v}\right)/\kappa^2$ and the second equality follows from $B^2(h) = (2B(h) - B(2h))/\kappa$.

Derivation of (3.1)

The Lagrangian $\mathcal{L}$ is given by

$$
\begin{aligned}
\mathcal{L} &= \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}\left(\frac{c(t)}{\Pi(t)}\right)^{1-\gamma} \mathrm{d}t\right] + y\left(A(0) - \mathbb{E}\left[\int_0^T m(t)c(t)\,\mathrm{d}t\right]\right) \\
&= \int_0^T \mathbb{E}\left[e^{-\delta t}\frac{1}{1-\gamma}\left(\frac{c(t)}{\Pi(t)}\right)^{1-\gamma} - ym(t)c(t)\right]\mathrm{d}t + yA(0).
\end{aligned}
\tag{A12}
$$

Here $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. The individual aims to maximize $e^{-\delta t}\frac{1}{1-\gamma}\left(\frac{c(t)}{\Pi(t)}\right)^{1-\gamma} - ym(t)c(t)$. The optimal consumption choice $c^*(t)$ satisfies the following first-order optimality condition:

$$
e^{-\delta t}\frac{1}{\Pi(t)}\left(\frac{c^*(t)}{\Pi(t)}\right)^{-\gamma} = ym(t). \tag{A13}
$$

After solving the first-order optimality condition, we obtain the following optimal consumption choice:

$$
c^*(t) = \Pi_t\left(e^{\delta t}y\Pi(t)m(t)\right)^{-\frac{1}{\gamma}}. \tag{A14}
$$

Substituting the expression for the consumer price index $\Pi(t) = \exp\left\{\int_0^t \pi(s)\,\mathrm{d}s\right\}$ and the stochastic discount factor $m(t) = \exp\left\{-\int_0^t \left(R(t) + \frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}t + \phi^\top\int_0^t \mathrm{d}W(t)\right\}$ into (A14), we arrive at (3.1).

Derivation of (3.4), (3.5) and (3.6)

Denote by $V^*(t)$ the market-consistent value at adult age t of current and future optimal consumption choices. We define $V^*(t)$ as follows:

$$
\begin{aligned}
V^*(t) &= \int_0^{T-t}\mathbb{E}_t\left[\frac{m(t+h)}{m(t)}c^*(t+h)\right]\mathrm{d}h \\
&= c^*(t)\int_0^{T-t}\mathbb{E}_t\left[\frac{m(t+h)}{m(t)}\frac{c^*(t+h)}{c^*(t)}\right]\mathrm{d}h = c^*(t)K^*(t),
\end{aligned}
\tag{A15}
$$

where $K^*(t)$ denotes the conversion factor at adult age t:

$$K^*(t) = \int_0^{T-t} \mathbb{E}_t \left[\frac{m(t+h)}{m(t)} \frac{c^*(t+h)}{c^*(t)} \right] \mathrm{d}h = \int_0^{T-t} \exp\left\{-d^*(t,h)h\right\} \mathrm{d}h. \tag{A16}$$

Here, $d^*(t,h)$ represents the market-consistent discount rate at adult age t for horizon $h \geq 0$. Straightforward computations show that

$$\begin{aligned} d^*(t,h) = \frac{1}{h} \Bigg[& \left(1 - \frac{1}{\gamma}\right) \int_0^h \left(r(t) + \kappa B(h) \left(\bar{r} - r(t)\right) + \frac{1}{2} \phi^\top \rho \phi \right) \mathrm{d}h \\ & - \frac{1}{2} \left(1 - \frac{1}{\gamma}\right)^2 \int_0^h \left(\phi_r - B(h)\sigma_r\right)^2 \mathrm{d}h \\ & - \left(1 - \frac{1}{\gamma}\right)^2 \rho_{r\pi} \int_0^h \left(\phi_r - B(h)\sigma_r\right) \phi_\pi \, \mathrm{d}h \\ & - \left(1 - \frac{1}{\gamma}\right)^2 \rho_{rS} \int_0^h \left(\phi_r - B(h)\sigma_r\right) \phi_S \, \mathrm{d}h \Bigg] \\ - \frac{1}{2} \left(1 - \frac{1}{\gamma}\right)^2 \phi_\pi^2 & - \frac{1}{2} \left(1 - \frac{1}{\gamma}\right)^2 \phi_S^2 - \left(1 - \frac{1}{\gamma}\right)^2 \rho_{\pi S} \phi_\pi \phi_S. \end{aligned} \tag{A17}$$

The quantity $\log V^*(t)$ evolves according to (this follows from (3.1), (A16) and (A17))

$$\begin{aligned} \mathrm{d}\log V^*(t) &= \mathrm{d}\log c^*(t) + \mathrm{d}\log K^*(t) \\ &= (\ldots)\,\mathrm{d}t - \left(\frac{1}{\gamma}\phi_r + D_{K^*}(t)\sigma_r \right) \mathrm{d}W_r(t) - \frac{1}{\gamma}\phi_\pi \, \mathrm{d}W_\pi(t) - \frac{1}{\gamma}\phi_S \, \mathrm{d}W_S(t). \end{aligned} \tag{A18}$$

Here, $D_{K^*}(t)$ represents the duration of the conversion factor:

$$D_{K^*}(t) = \left(1 - \frac{1}{\gamma}\right) \int_0^{T-t} \frac{V^*(t,h)}{V^*(t)} B(h) \, \mathrm{d}h, \tag{A19}$$

where $V^*(t,h) = c^*(t) \exp\left\{-d^*(t,h)h\right\}$.

Log total wealth evolves according to:

$$\begin{aligned} \mathrm{d}\log A(t) = (\ldots)\,\mathrm{d}t & - \left[\omega_N(t)B\left(h_N\right) + \omega_I(t)B\left(h_I\right)\right] \sigma_r \mathrm{d}W_r(t) \\ & - \omega_N(t)C\left(h_N\right)\sigma_\pi \mathrm{d}W_\pi(t) + \omega_S(t)\sigma_S \mathrm{d}W_S(t). \end{aligned} \tag{A20}$$

Comparing (A20) with (A18), we arrive at (3.4), (3.5) and (3.6).

Derivation of (4.1)

Denote by $L(t)$ and $H(t)$ labor income and human capital at adult age t, respectively. We define human capital as follows:

$$H(t) = \int_0^{T-t} H(t,h)\,\mathrm{d}h, \tag{A21}$$

where

$$H(t,h) = \mathbb{E}_t\left[\frac{m(t+h)}{m(t)}L(t+h)\right] \tag{A22}$$

with

$$L(t,h) = \begin{cases} \Pi(t+h) & \text{if } t+h < T_R \\ s\cdot\Pi(t+h) & \text{if } t+h \geq T_R. \end{cases} \tag{A23}$$

Here, T_R denotes the age at which the individual retires and $0 \leq s \leq 1$ represents the pension payment from the government (expressed as a share of the consumer price index).

Straightforward computations show

$$\mathrm{d}H(t) = \left(R(t) - \lambda_r\sigma_r D_H(t)\right)H(t)\,\mathrm{d}t - D_H(t)\sigma_r H(t)\,\mathrm{d}W_r(t) - L(t)\,\mathrm{d}t, \tag{A24}$$

where

$$D_H(t) = \int_0^{T-t}\frac{H(t,h)}{H(t)}B(h)\,\mathrm{d}h \tag{A25}$$

denotes the duration of human capital.

Financial wealth $F(t)$ evolves as follows:

$$\begin{aligned} \mathrm{d}F(t) = (\ldots)\,\mathrm{d}t &- \left[\widehat{\omega}_N(t)B\left(h_N\right) + \widehat{\omega}_I(t)B\left(h_I\right)\right]\sigma_r F(t)\mathrm{d}W_r(t) \\ &- \widehat{\omega}_N(t)C\left(h_N\right)\sigma_\pi F(t)\mathrm{d}W_\pi(t) + \widehat{\omega}_S(t)\sigma_S F(t)\mathrm{d}W_S(t). \end{aligned} \tag{A26}$$

Hence, total wealth $A(t) = H(t) + F(t)$ satisfies

$$\begin{aligned} \mathrm{d}A(t) &= \mathrm{d}H(t) + \mathrm{d}F(t) \\ &= (\dots)\,\mathrm{d}t - \left[\widehat{\omega}_N(t)B\left(h_N\right)\frac{F(t)}{A(t)} + \widehat{\omega}_I(t)B\left(h_I\right)\frac{F(t)}{A(t)} + D_H(t)\frac{H(t)}{A(t)}\right]\sigma_r A(t)\mathrm{d}W_r(t) \\ &\quad - \widehat{\omega}_N(t)C\left(h_N\right)\frac{F(t)}{A(t)}\sigma_\pi A(t)\mathrm{d}W_\pi(t) + \widehat{\omega}_S(t)\sigma_S\frac{F(t)}{A(t)}A(t)\mathrm{d}W_S(t). \end{aligned} \tag{A27}$$

Comparing (2.9) with (A27), we arrive at (4.1).

Appendix B: Numerical Solution Technique

We determine the optimal consumption and portfolio policies using numerical backward induction. We start by discretizing both time and the state space. We first specify discrete points in the state space, called grid points, for the final time period. For each grid point, we determine the optimal consumption choice, the optimal portfolio choice and the level of the value function. For the final period these values are trivial, since the individual simply consumes any remaining wealth. We then move one period back in time. We then first derive the optimal portfolio decision for wealth after consumption for all points on the state space grid. Subsequently we determine optimal consumption using the endogonous grid method proposed by Carroll (2006). This allows us to exploit the analytical first order condition to the optimal consumption problem, which means we do not need to numerically search for the solution. Finding the optimal portfolio weights and optimal consumption does requires us to evaluate the expected value of the utility function next period. We do so by numerical integration over the state space using Gaussian Quadrature. Whenever the integration algorithm requires points that are not on the grid, we use an interpolation technique. In particular, we linearly interpolate a certainty equivalent measure: the certain and flat level of consumption that would deliver the level of utility in the grid point. We then convert the interpolated certainty equivalent back to utility terms. The idea behind this approach is that in the unconstrained problem, this certainty equivalent would be a linear function of the endogonous state (wealth) and hence our procedure would yield the exact utility value for wealth levels not on our grid.

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