Consumption and Portfolio Choice under Internal Multiplicative Habit Formation*

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Abstract

This paper explores the optimal consumption and investment behavior of an individual who derives utility from the ratio between his consumption and an endogenous habit. We obtain closed-form policies under general utility functionals and stochastic investment opportunities, by developing a non-trivial linearization to the budget constraint. This enables us to explicitly characterize how habit formation affects the marginal propensity to consume and optimal stock-bond investments. We also show that in a setting which combines habit formation with Epstein-Zin utility, consumption no longer grows at unrealistically high rates at high ages and investments in risky assets decrease.

JEL classification: D15, D81, G11, G13.

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1 Introduction

The internal habit formation literature, in which individuals draw utility from consumption relative to an endogenous habit, can be divided along two main model specifications that have been widely used in economics and finance: additive habits[1] (Constantinides (1990)) and multiplicative habits (Abel (1990)).[2] While both models are appealing from a prescriptive and a descriptive point of view, the latter model specification, also referred to as the ratio internal habit model, is advocated in particular by Carroll (2000) and Fuhrer (2000). Just like the additive internal habit model, the ratio internal habit model can be rationalized (see, e.g., Crawford (2010) and references therein) while, at the same time, it can account for the observed degree of excess smoothness in consumption. Contrary to the additive internal habit model, the ratio internal habit model does not require an artificial constraint on the individual's initial wealth position or on the habit dynamics to avoid negative infinite utility.[3] Multiplicative internal habits play a central role in this paper.

A main ingredient of optimal consumption and portfolio choice problems is the temporal structure of preferences. The ratio internal habit model implies time-inseparability of preferences, but maintains a time additive structure in terms of relative consumption.[4] As is well-known, an additive structure where utility is additive over time and states of nature implies that the elasticity of intertemporal substitution (EIS) and risk aversion are linked. This is analytically convenient yet fairly restrictive. Therefore, we study multiplicative internal habits also under Epstein-Zin utility (Epstein and Zin (1989)).[5] This preference model decouples the EIS from risk aversion and has been widely used in the consumption and portfolio choice literature.[6]

In this paper we develop a closed-form approach to solve consumption and portfolio choice problems involving multiplicative internal habits. To analyze how multiplicative internal habit formation affects the conventional wisdom on optimal consumption and

[1]Additive habits are also referred to as subtractive habits, linear habits, or the difference habit model.

[2]Some authors assume that habits are external rather than internal; see, e.g., Abel (1990)'s catching-up-with-the-Joneses specification, Campbell and Cochrane (1999), and Chan and Kogan (2002).

[3]See, e.g., Carroll (2000) and Munk (2008) for a discussion of this point.

[4]Henceforth, relative consumption is defined as the ratio between consumption and the endogenous habit level.

[5]Strictly speaking, we consider stochastic differential utility (SDU), which arises as a continuous-time limit of Epstein-Zin utility; see Duffie and Epstein (1992). See also Kraft and Seifried (2014) who show that Epstein-Zin utility converges to SDU.

[6]See, e.g., Campbell and Viceira (1999), Campbell, Cocco, Gomes, Maenhout, and Viceira (2001), Chacko and Viceira (2005), and Bhamra and Uppal (2006).

investment dynamics, we apply our general approach to three important cases: a base case with multiplicative internal habits, additive utility in terms of relative consumption, and constant investment opportunities; an extension of the base case with stochastic investment opportunities involving stochastic interest rates; and a case which combines multiplicative internal habits with Epstein-Zin utility.

In a nutshell, our approach consists of first applying a change of variables, redefining consumption in relative terms, and next pursuing a suitable pathwise linearization of the static budget constraint around the endogenous habit level.[7] Our approximation approach transforms consumption and portfolio choice problems with multiplicative internal habits into approximate consumption and portfolio choice problems *without* habits. This enables us to obtain closed-form approximate solutions to a variety of consumption and portfolio choice problems with multiplicative internal habit formation under general utility functionals and stochastic investment opportunities.

We can summarize our three main findings as follows. First, we characterize in explicit closed-form how a habit-forming individual in the base-line model adjusts both his current consumption level and future growth rates of consumption after a stock return shock. While consumption is well-known to be excessively smooth under the ratio internal habit model, an explicit closed-form characterization of the shock absorbing mechanism is new. We show that the features of the marginal propensity to consume (i.e., shock absorbing mechanism) and investment strategy are determined by two factors: the degree of relative risk aversion and the strength of habit persistence. These factors not only have clear economic interpretations themselves but also induce clearly interpretable implications for the optimal consumption and portfolio decisions: the degree of relative risk aversion determines how large the impact of a stock return shock is on the individual's current consumption level,[8] and the strength of habit persistence determines the horizon-dependent impact of a stock return shock on future growth rates of consumption.[9,10] By contrast, under conventional constant relative risk

[7]Linearization of the static budget constraint is not uncommon in the economics literature; see, in a different context, e.g., Campbell and Mankiw (1991), and Fuhrer (2000).

[8]In particular, we show that a stock return shock has a smaller impact on the current consumption level of a highly risk-averse individual than on that of a weakly risk-averse individual.

[9]We find that the more persistent the habit level is, the larger the impact of a stock return shock on future growth rates of consumption will be.

[10]We argue that the optimal policies provide a preference-based justification for the existence of annuity products in which surpluses earned in good years support benefit payouts in bad years. Such annuity products have been analyzed by e.g., Guillén, Jørgensen, and Nielsen (2006), Jørgensen and Linnemann (2012), Guillén, Nielsen, Pérez-Marín, and Petersen (2013), Maurer, Rogalla, and Siegelin (2013a), Linnemann, Bruhn, and Steffensen (2014), and Maurer, Mitchell, Rogalla, and Siegelin (2016).

aversion (CRRA) utility, which has become a main benchmark since Merton (1969), the impact of a stock return shock on consumption is uniformly distributed over time: it does not depend on the time distance between the occurrence of the shock and the date of consumption. We also find that an increase in habit persistence leads to a riskier investment strategy while leaving the year-on-year consumption volatility unaffected.[11] As a result, current consumption of a habit-forming individual is less volatile than his underlying investment portfolio. Furthermore, we show that a habit-forming individual implements a life-cycle investment strategy that is nearly independent of the state of the economy (especially at high ages) and depends only on age.[12] Contrary to under conventional CRRA utility, we do not need human capital to justify a life-cycle investment strategy.[13]

Second, in an extension of our base-line model that allows for stochastic interest rates and stock-bond investments, we find that the interest rate duration of the optimal hedging bond portfolio is hump shaped over the life cycle, which contradicts the conventional wisdom that the duration of the optimal hedging bond portfolio is decreasing with age. Two counteracting forces determine the life-cycle pattern of the duration of the optimal hedging bond portfolio. On the one hand, the impact of an interest rate shock on the price of future consumption is larger the younger the individual is. This effect causes the duration of the optimal hedging bond portfolio to decrease with age and is familiar from Brennan and Xia (2002) (see also Merton (2014)). On the other hand, we find a new effect that causes the duration of the optimal hedging bond portfolio to increase with age. We can explain this effect by the fact that a habit-forming individual is less willing to substitute consumption over time as he grows older. Intuitively, as the individual grows older, the duration of remaining lifetime consumption declines, and hence the current habit level determines to a greater extent future consumption levels.[14]

A general feature of many habit formation models (including our base-line model)

[11]This finding stands in sharp contrast to standard unit-linked insurance products and traditional drawdown strategies in which a more aggressive portfolio strategy directly translates into a higher year-on-year consumption volatility. See, e.g., Dus, Maurer, and Mitchell (2005), Horneff, Maurer, Mitchell, and Dus (2008), and Maurer, Mitchell, Rogalla, and Kartashov (2013b) for a description of these products.

[12]More specifically, the individual in our base-line model lowers the share of his portfolio invested in the risky stock as he becomes older. Indeed, the available time to adjust current and future consumption levels in response to a stock return shock declines with age.

[13]For the classical implications of human capital on the optimal portfolio allocation, see Bodie, Merton, and Samuelson (1992) and Cocco, Gomes, and Maenhout (2005).

[14]The second effect may explain why not many young individuals include long-term bonds in their investment portfolios; see Morningstar (2017) for the investment behavior of long-term investors.

is that median consumption grows at unrealistically high rates (especially at high ages) except when the time discount rate is excessive. We therefore also analyze a model that combines Epstein-Zin utility with multiplicative internal habits.[15] Our third main finding is, then, that in this setting that decouples the EIS from risk aversion, habit formation does not necessarily lead to unrealistically high unconditional median growth rates of consumption at the end of life, even when the time discount rate is moderate. Furthermore, wealth accumulation is substantially lower under this extended model than under the base-line model. Hence, an individual whose preferences combine multiplicative internal habit formation with Epstein-Zin utility invests less wealth in the stock market compared to an individual without Epstein-Zin utility.

The endogenous nature of the habit in internal habit formation models substantially complicates the analysis of optimal consumption and portfolio policies and asset pricing problems. In important work, Schroder and Skiadas (2002) show how to transform models with *additive* internal habits into models without habit formation, enabling closed-form solutions to a wide range of asset pricing problems involving additive internal habits.[16] Conversely, their approach allows to translate solutions to familiar consumption and portfolio choice problems under general utility functionals into solutions to corresponding problems exhibiting additive internal habit formation. So far, however, internal habit formation models with *multiplicative* habits cannot be solved analytically. Thus, analysis of the appealing ratio internal habit model necessarily resorted to numerical methods to obtain solutions, impeding their applicability.

We obtain closed-form solutions to consumption and portfolio choice problems featuring multiplicative internal habits based on developing a pathwise approximation to the budget constraint. Our numerical results show that the approximation error, when measured in terms of the relative decline in certainty equivalent consumption, is typically less than 1%, and that the explicit optimal policies to the approximate problems closely mimic the numerically evaluated optimal policies to the original problems. Having closed-form solutions has three key advantages: they reveal the roles played by the various model parameters, they are readily amenable to comparative statics analysis, and they facilitate the implementation of the optimal consumption and investment policies.

[15] The closest to the current paper in this respect is Schroder and Skiadas (1999) who analytically studied Epstein-Zin utility but did not consider multiplicative internal habits.

[16] See, e.g., Van Bilsen, Laeven, and Nijman (2017) who employ Schroder and Skiadas (2002) to explicitly derive the optimal consumption and portfolio policies under loss aversion and endogenous updating of the reference level—two key features of prospect theory (Tversky and Kahneman (1992)).

We note that the problem of optimal consumption and portfolio choice over the life cycle has intrigued many authors since the seminal work of Mossin (1968), Merton (1969, 1971), and Samuelson (1969). Their work has been extended along many dimensions.[17] Many life-cycle consumption and portfolio choice papers assume a standard preference model; that is, decision makers' preferences are described by time-additive CRRA utility or Epstein-Zin Constant EIS-CRRA utility. While these standard preference models satisfy a set of normatively compelling axioms, their ability to describe how people actually make decisions under risk is known to be limited. Furthermore, their predictions fail to explain well-documented facts about actual consumption and portfolio behavior such as the excess smoothness of consumption. The shortcomings of standard preference models have inspired many researchers to develop alternative theories for decision-making under risk,[18] including habit formation.

Several authors have explored the implications of these alternative preference theories for optimal investment decisions or intertemporal consumption behavior.[19] Most relevant to our base-line model are Detemple and Zapatero (1991, 1992), Schroder and Skiadas (2002), Bodie, Detemple, Otruba, and Walter (2004) and Munk (2008) who analyze the optimal consumption and investment behavior of an individual who derives utility from the difference between consumption and an internal habit level, rather than some ratio of these as we do. Contrary to under the ratio habit model, the optimal consumption choice implied by the difference habit model exceeds the habit level in each economic scenario. This so-called addictive behavior of consumption is criticized theoretically e.g., by Chapman (1998) and Carroll (2000), and arguably at odds with empirical evidence.[20] Finally, the ratio habit model has been employed in other papers to analyze monetary policy (Fuhrer (2000)), asset prices with an external habit (Abel (1999), Chan and Kogan (2002) and Gómez, Priestley, and Zapatero (2009)) and an

[17]For instance, to accommodate time-varying investment opportunities (see, e.g., Campbell et al. (2001), Wachter (2002), Chacko and Viceira (2005), Liu (2007), and Laeven and Stadje (2014)); uncertain labor income (see, e.g., Viceira (2001), Cocco et al. (2005), and Gomes and Michaelides (2005)); housing costs (see, e.g., Cocco (2005), and Yao and Zhang (2005)); and unexpected health expenditures (see, e.g., Edwards (2008)).

[18]Among the most notable alternatives are prospect theory (Kahneman and Tversky (1979), and Tversky and Kahneman (1992)), regret theory (Loomes and Sugden (1982), Bell (1982, 1983), Sugden (1993), and Quiggin (1994)), disappointment (aversion) theory (Bell (1985), Loomes and Sugden (1986), and Gul (1991)), and habit formation (Abel (1990), Constantinides (1990) and Sundaresan (1989)).

[19]See, e.g., Bowman, Minehart, and Rabin (1999), Berkelaar, Kouwenberg, and Post (2004), Ang, Bekaert, and Lui (2005), Muermann, Mitchell, and Volkman (2006), Guasoni, Huberman, and Ren (2015), Pagel (2017) and Van Bilsen et al. (2017).

[20]For instance, Crossley, Low, and O'Dea (2013) show that consumption levels declined significantly during recent recessions, contradicting the addictive property of consumption.

internal habit (Smith and Zhang (2007)), macroeconomic growth (Carroll, Overland, and Weil (1997), Carroll, Overland, and Weil (2000) and Carroll (2000)), and portfolio choice with uninsurable labor income risk (Gomes and Michaelides (2003)).

2 Model

2.1 Asset Prices, Pricing Kernel and Budget Constraint

Denote by $T > 0$ a fixed terminal time. We represent the randomness in the economy by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which is defined a standard N-dimensional Brownian motion $\{W_t\}_{0\leq t\leq T}$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ is the augmentation under $\mathbb{P}$ of the natural filtration generated by the standard Brownian motion $\{W_t\}_{0\leq t\leq T}$. Throughout, (in)equalities between random variables hold $\mathbb{P}$-almost surely.

We consider a financial market consisting of an instantaneously risk-free asset and N risky assets. Trading takes place continuously over $[0, T]$. The price of the risk-free asset, B_t, satisfies

$$\frac{\mathrm{d}B_t}{B_t} = r_t\,\mathrm{d}t, \quad B_0 = 1. \tag{1}$$

We assume that the scalar-valued risk-free rate process, $\{r_t\}_{0\leq t\leq T}$, is $\mathcal{F}_t$-progressively measurable and satisfies $\int_0^T |r_t|\,\mathrm{d}t < \infty$. The N-dimensional vector of risky asset prices, S_t, obeys the following stochastic differential equation:

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t, \quad S_0 = \mathbf{1}_N. \tag{2}$$

Here, $\mathbf{1}_N$ represents an N-dimensional vector consisting of all ones. We assume that the N-dimensional mean rate of return process, $\{\mu_t\}_{0\leq t\leq T}$, and the $(N \times N)$-matrix-valued volatility process, $\{\sigma_t\}_{0\leq t\leq T}$, are $\mathcal{F}_t$-progressively measurable and satisfy $\int_0^T ||\mu_t||\,\mathrm{d}t < \infty$ and $\sum_{i=1}^N \sum_{j=1}^N \int_0^T (\sigma_t)_{ij}^2\,\mathrm{d}t < \infty$, respectively. We impose the following additional condition on σ_t. For some $\epsilon > 0$,

$$\zeta^\top \sigma_t \sigma_t^\top \zeta \geq \epsilon ||\zeta||^2, \quad \text{for all } \zeta \in \mathbb{R}^N, \tag{3}$$

where $\top$ denotes the transpose sign. Condition (3) implies in particular that σ_t is invertible.

The $\mathcal{F}_t$-progressively measurable market price of risk process, $\{\lambda_t\}_{0\leq t\leq T}$, satisfies

$$\sigma_t \lambda_t = \mu_t - r_t \mathbf{1}_N. \tag{4}$$

The unique positive-valued state price density process, $\{M_t\}_{0\leq t\leq T}$, is given by (see, e.g., Karatzas and Shreve (1998)):

$$M_t = \exp\left\{-\int_0^t r_s \, \mathrm{d}s - \int_0^t \lambda_s^\top \, \mathrm{d}W_s - \frac{1}{2}\int_0^t ||\lambda_s||^2 \, \mathrm{d}s\right\}. \tag{5}$$

The economy consists of a single individual endowed with initial wealth $A_0 \geq 0$. This individual chooses an $\mathcal{F}_t$-progressively measurable N-dimensional portfolio process $\{\pi_t\}_{0\leq t\leq T}$ (representing the dollar amounts invested in the N risky assets) and an $\mathcal{F}_t$-progressively measurable consumption process $\{c_t\}_{0\leq t\leq T}$ so as to maximize lifetime utility. We impose the following integrability conditions on the portfolio and consumption processes:

$$\int_0^T \pi_t^\top \sigma_t \sigma_t^\top \pi_t \, \mathrm{d}t < \infty, \quad \int_0^T \left|\pi_t \left(\mu_t - r_t \mathbf{1}_N\right)\right| \mathrm{d}t < \infty, \quad \mathbb{E}\left[\int_0^T |c_t|^r \, \mathrm{d}t\right] < \infty \; \forall\, r \in \mathbb{R}. \tag{6}$$

The wealth process, $\{A_t\}_{0\leq t\leq T}$, satisfies the following dynamic budget constraint:

$$\mathrm{d}A_t = \left(r_t A_t + \pi_t^\top \sigma_t \lambda_t - c_t\right) \mathrm{d}t + \pi_t^\top \sigma_t \, \mathrm{d}W_t, \quad A_0 \geq 0 \text{ given.} \tag{7}$$

We call a consumption-portfolio strategy $\{c_t, \pi_t\}_{0\leq t\leq T}$ admissible if the associated wealth process is positive.

2.2 Habit Level

Denote by h_t the individual's habit level at time t. Following Kozicki and Tinsley (2002) and Corrado and Holly (2011), we assume that the log habit level $\log h_t$ satisfies the following dynamic equation:[21]

$$\mathrm{d}\log h_t = (\beta \log c_t - \alpha \log h_t)\, \mathrm{d}t, \quad \log h_0 = 0. \tag{8}$$

[21] The *log* habit level $\log h_t$ is additive, i.e., linear, in past levels of log consumption. Corrado and Holly (2011) show that for the ratio internal habit model, the habit specification (8) (see also (10)) is more desirable than an arithmetic habit specification in which the habit level h_t itself is additive in past levels of consumption.

We normalize the initial log habit $\log h_0$ to zero, i.e., h_0 equals unity. The preference parameter $\alpha \geq 0$ represents the rate at which the log habit level exponentially depreciates. When α is small, the log habit level exhibits a high degree of memory. The preference parameter $\beta \geq 0$ models the relative importance between the initial habit level and the individual's past consumption choices. When β is large, the individual's past consumption choices are relatively important. We impose the following restriction on the individual's preference parameters:

$$\alpha \geq \beta. \tag{9}$$

The parameter restriction (9) prevents the individual's habit level from growing exponentially over time; see Eqn. (15) below. In the special case where $\beta = 0$, the habit level is exogenously given.

Finally, we note that we can write the log habit level $\log h_t$ as a weighted sum of the individual's log past consumption choices:

$$\log h_t = \beta \int_0^t \exp\{-\alpha(t-s)\} \log c_s \, \mathrm{d}s. \tag{10}$$

2.3 Dynamic Optimization Problem

Let $U(c/h) \in \mathbb{R} \cup \{-\infty\}$ be the individual's lifetime utility derived from the process $c/h = \{c_t/h_t\}_{0 \leq t \leq T}$ representing the ratio between consumption and the habit level. We place no restrictions on U. The individual now faces the following dynamic optimization problem over admissible consumption-portfolio strategies $\{c_t, \pi_t\}_{0 \leq t \leq T}$:

$$\begin{aligned} \max_{c_t, \pi_t : 0 \leq t \leq T} \quad & U\left(\frac{c}{h}\right) \\ \text{s.t.} \quad & \mathrm{d}A_t = \left(r_t A_t + \pi_t^\top \sigma_t \lambda_t - c_t\right) \mathrm{d}t + \pi_t^\top \sigma_t \, \mathrm{d}W_t, \\ & \mathrm{d}\log h_t = (\beta \log c_t - \alpha \log h_t)\, \mathrm{d}t, \end{aligned} \tag{11}$$

with $A_0 \geq 0$ given.

Section 3 presents a solution technique for analytically solving (11) based on developing a pathwise linearization to the individual's budget constraint.

3 Solution Method

3.1 An Equivalent Problem

We can, by virtue of the martingale approach (Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989, 1991)), transform the individual's dynamic optimization problem (11) into the following equivalent static variational problem:

$$\begin{aligned} \max_{c_t:0\leq t\leq T} \quad & U\left(\frac{c}{h}\right) \\ \text{s.t.} \quad & \mathbb{E}\left[\int_0^T M_t c_t \,\mathrm{d}t\right] \leq A_0, \\ & \mathrm{d}\log h_t = (\beta \log c_t - \alpha \log h_t)\,\mathrm{d}t, \end{aligned} \tag{12}$$

where M_t is given by (5). After the optimal consumption choice c_t^{opt} has been determined, one can determine the optimal portfolio choice π_t^{opt} using hedging arguments.

3.2 A Change of Variable Transformation

Denote by $\widehat{c}_t$ the ratio between the individual's consumption choice and his habit level; that is,

$$\widehat{c}_t = \frac{c_t}{h_t}. \tag{13}$$

We can express the dynamics of the log habit level in terms of the individual's log relative consumption choice $\log \widehat{c}_t = \log(c_t/h_t)$ as follows:[22]

$$\mathrm{d}\log h_t = (\beta \log \widehat{c}_t - [\alpha - \beta] \log h_t)\,\mathrm{d}t. \tag{14}$$

Hence, the individual's log habit level $\log h_t$ is explicitly given by

$$\log h_t = \beta \int_0^t \exp\{-(\alpha - \beta)(t - s)\} \log \widehat{c}_s \,\mathrm{d}s. \tag{15}$$

Eqn. (15) shows that, as a result of the parameter restriction $\alpha \geq \beta$ (see (9)), the individual's habit level is prevented from growing exponentially over time.

We can thus rewrite the individual's static optimization problem (12) in terms of

[22]The dynamics of the log habit level (14) follow from substituting $\log c_t = \log h_t + \log \widehat{c}_t$ into (8).

$\widehat{c}_t = c_t/h_t$ yielding the following equivalent problem:

$$\begin{aligned} \max_{\widehat{c}_t:0\leq t\leq T} \quad & U(\widehat{c}) \\ \text{s.t.} \quad & \mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right] \leq A_0, \\ & \mathrm{d}\log h_t = (\beta \log \widehat{c}_t - [\alpha - \beta]\log h_t)\,\mathrm{d}t. \end{aligned} \tag{16}$$

We then obtain the optimal consumption choice c_t^{opt} from the optimal relative consumption choice $\widehat{c}_t^{\text{opt}}$ as follows:[23]

$$c_t^{\text{opt}} = h_t^{\text{opt}} \widehat{c}_t^{\text{opt}}. \tag{17}$$

To solve the individual's static optimization problem (12), we can thus restrict ourselves to solving (16). In applications, it is still typically impossible to solve the individual's static optimization problem (16) analytically. The reason for this is that the new static budget constraint in (16) depends non-linearly on the individual's relative consumption choices.[24] Section 3.3 develops a pathwise linearization for the new static budget constraint in (16). After applying this linearization, we are able to obtain analytical closed-form expressions for the individual's consumption and investment policies in a wide range of interesting cases.

3.3 Linearization of the New Static Budget Constraint

This section presents a linear approximation to the left-hand side of the new static budget constraint in (16) around the relative consumption trajectory $\{\widehat{c}_t\}_{0\leq t\leq T} = 1$.[25] Consumption c_t is thus approximated around the endogenous habit level h_t. The key insight here is that, because the habit level is determined endogenously by the individual's own weighted past consumption choices, it tracks consumption. As a consequence, the habit level is a natural candidate around which to apply the approximation. This yields a 'pathwise approximation'. Section 7 explores in detail the approximation error induced by applying our pathwise approximation to the new static budget constraint in (16).

[23] We can determine h_t^{opt} by substituting the optimal past relative consumption choices $\widehat{c}_s^{\text{opt}}$ $(s \leq t)$ into (15).

[24] Indeed, substitution of the habit level h_t (see (15)) into the budget constraint in (16) shows that the new static budget constraint in (16) is non-linear in the individual's relative consumption choices.

[25] The Appendix considers the more general case in which the budget constraint is approximated around the relative consumption trajectory $\{\widehat{c}_t\}_{0\leq t\leq T} = x$ for some positive x.

The numerical results reveal that the approximation error is typically less than 1% in terms of relative decline in certainty equivalent consumption and that our closed-form approximated strategies closely mimic the genuinely optimal (but numerically evaluated) strategies.

Appendix A proves the following theorem.

Theorem 3.1. *Consider an individual who aims to solve the optimization problem* (12)*. This problem is equivalent to the following simpler problem up to a first-order approximation of the static budget constraint:*

$$\begin{aligned} \max_{\widehat{c}_t:0\leq t\leq T} \quad & U\left(\widehat{c}\,\right) \\ s.t. \quad & \mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t \,\mathrm{d}t\right] \leq \widehat{A}_0. \end{aligned} \tag{18}$$

Here,

$$\widehat{M}_t = M_t\left(1+\beta P_t\right) \tag{19}$$

with P_t denoting the price at time t of a bond that pays the coupon process $\left\{e^{-(\alpha-\beta)(s-t)}\right\}_{s\geq t}$, i.e.,

$$P_t = \mathbb{E}_t\left[\int_t^T \frac{M_s}{M_t} e^{-(\alpha-\beta)(s-t)}\,\mathrm{d}s\right]. \tag{20}$$

The quantity $\widehat{A}_0$ denotes the individual's initial wealth associated with the approximate problem (18)*. We determine the individual's initial wealth $\widehat{A}_0$ such that the approximate optimal consumption strategy $\{c_t^*\}_{0\leq t\leq T} = \{h_t^* \widehat{c}_t^{\,*}\}_{0\leq t\leq T}$ is budget-feasible.*[26]

The relative consumption choice $\widehat{c}_t^{\,*}$ solving (18) is an approximation to the optimal relative consumption choice $\widehat{c}_t^{\,\mathrm{opt}}$. Note that the endogenous habit level h_t now does not appear in (18), thanks to the change of variable transformation and, crucially, our pathwise linearization of the static budget constraint.

Remark 1. *Using a suitable transformation, Schroder and Skiadas (2002) translate models with an internal habit and utility expressed as a function of the difference between consumption and a habit level into models without habit formation. Interestingly and quite surprisingly (to us), the transformed state-price density process* (19) *and hence the transformed problem* (18) *are identical to the transformed*

[26]Here, h_t^* denotes the habit level at time t implied by substituting the approximate optimal past relative consumption choices $\widehat{c}_s^{\,*}$ $(s \leq t)$ into (15).

counterparts in Schroder and Skiadas (2002), the difference being that $\widehat{c}_t$ represents surplus consumption $c_t - h_t$ in their framework while it represents relative consumption c_t/h_t in our framework. In their setting, the original budget constraint is equivalent to the budget constraint in the transformed problem:

$$\mathbb{E}\left[\int_0^T M_t c_t \,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \widehat{M}_t \left(c_t - h_t\right) \mathrm{d}t\right] + K_1, \tag{21}$$

while in our setting the new budget constraint first-order approximates the original budget constraint:

$$\mathbb{E}\left[\int_0^T M_t c_t \,\mathrm{d}t\right] \approx \mathbb{E}\left[\int_0^T \widehat{M}_t \frac{c_t}{h_t} \,\mathrm{d}t\right] + K_2. \tag{22}$$

Here, K_1 and K_2 are constants that are irrelevant in determining the first-order optimality conditions. Note also that the interpretation of the parameters α and β is different in the two papers as we consider the dynamics of the log habit level $\log h_t$ while they consider the dynamics of the habit level h_t.

4 Ratio Internal Habit Model

This section assumes that lifetime utility is defined as follows:

$$U\left(\frac{c}{h}\right) = \mathbb{E}\left[\int_0^T e^{-\delta t} \frac{1}{1-\gamma} \left(\frac{c_t}{h_t}\right)^{1-\gamma} \mathrm{d}t\right], \tag{23}$$

with h_t given by (10). Here, $\mathbb{E}$ represents the unconditional expectation, $\delta \geq 0$ stands for the individual's subjective rate of time preference, and $\gamma > 0$ denotes the individual's coefficient of relative risk aversion. Specification (23) corresponds to the habit formation model proposed by Abel (1990).

In (23), relative risk aversion is constant. Several authors explore the implications of the difference internal habit model in which relative risk aversion is not constant but rather depends on surplus consumption $c_t - h_t$. As a result, the optimal strategies under the difference internal habit model are considerably different from the optimal strategies under the ratio internal habit model. In particular, the portfolio strategy of an individual whose preferences are represented by the difference internal habit model heavily depends on the individual's endogenous habit level. In our model, by contrast, the portfolio strategy is nearly state-independent; see Section 4.5 for more details.

4.1 Optimal Consumption Choice

Theorem 4.1 presents the (approximate) optimal consumption choice c_t^*.

Theorem 4.1. *Consider an individual with lifetime utility* (23) *and habit formation process* (8) *who solves the consumption and portfolio choice problem* (18)*. Denote by h_t^* the habit level at time t implied by substituting the (approximate) optimal past relative consumption choices $\widehat{c}_s^*$ $(s \leq t)$ into* (15)*, and by y the Lagrange multiplier associated with the static budget constraint in* (18)*. Then the (approximate) optimal consumption choice c_t^* is given by*

$$c_t^* = h_t^* \left(y e^{\delta t} \widehat{M}_t \right)^{-\frac{1}{\gamma}}. \tag{24}$$

The Lagrange multiplier $y \geq 0$ is determined such that the individual's original budget constraint holds with equality.

Note that (24) is exactly equal to c_t^{opt} in case $\alpha = \beta = 0$.

4.2 Sensitivity and Volatility of Future Consumption

In the remainder of this section, we assume constant investment opportunities (i.e., $r_t = r$, $\mu_t = \mu$, $\sigma_t = \sigma$ and $\lambda_t = \lambda$ for all t) and only one risky stock. We set the risk-free interest rate r at 1%, the equity risk premium $e = \mu - r$ at 4%, and the stock return volatility σ at 20%. These parameter values are the same as those used in Gomes, Kotlikoff, and Viceira (2008). The rate of time preference δ is assumed to be equal to 3%.[27]

Denote by q_{t-s} the sensitivity of log consumption, $\log c_t^*$, to a past stock return shock, $\sigma \, \mathrm{d}W_s$ $(s \leq t)$. We find the explicit closed-form expression (see Appendix A)[28]

$$q_{t-s} = \frac{\lambda}{\gamma \sigma} Q_{t-s}, \tag{25}$$

with

$$Q_{t-s} = 1 + \frac{\beta}{\alpha - \beta} \left[1 - \exp\left\{ -(\alpha - \beta)(t-s) \right\} \right]. \tag{26}$$

The sensitivity q_{t-s}, dictating the optimal shock absorbing mechanism in analytical form, depends on the time distance between the date at which the stock return shock occurs (i.e.,

[27]Samwick (1998) finds that the median rates of time preference for US households are between 3% and 4%.

[28]We note that if $\alpha = \beta$, then (26) reduces to $Q_{t-s} = 1 + \beta(t-s)$.

time s) and the date of consumption (i.e., time $t > s$). In particular, a closer inspection of (25) reveals that q_h increases with the time distance (or horizon) h: a current stock return shock has a smaller impact on log consumption in the near future (i.e., small h) than on log consumption in the distant future (i.e., large h). In case the individual exhibits conventional constant relative risk aversion (CRRA) utility (henceforth referred to as a CRRA individual), which has become the main benchmark since Merton (1969), the sensitivity q_h is independent of the horizon h.

Our utility framework thus provides a preference-based justification for the existence of annuity products in which current stock return shocks are not fully reflected into current annuity payouts. These products work as follows (see, e.g., Guillén et al. (2006), Linnemann et al. (2014), and Maurer et al. (2016)). In the case of a positive investment return, the annuity payout will go up by less than the realized return. The remaining investment gains will be added to a reserve fund. In the case of a negative investment return, the annuity payout will be protected and will decrease by a lower percentage than the realized return. This 'payout protection' will be paid from the reserve fund. The just described mechanism results in an excessively smooth payout stream.

The individual's preference parameters γ, α and β have clearly interpretable implications for the individual's optimal consumption choice. We find that a current stock return shock $\sigma\,\mathrm{d}W_t$ has a smaller impact on the current consumption level of a highly risk-averse individual (i.e., high γ) than on that of a weakly risk-averse individual (i.e., low γ). Indeed, a highly risky-averse individual is more risk averse to year-on-year fluctuations in current consumption than a weakly risk-averse individual. The coefficients β and $\widehat{\alpha} = \alpha - \beta$, which measure the degree of habit persistence, determine the impact of a current stock return shock on the future growth rates of (median) consumption. If the individual's preferences exhibit a large degree of habit persistence (i.e., β is large and $\widehat{\alpha}$ is close to zero), a current stock return shock will have a relatively large impact on future growth rates of consumption: the individual adjusts the future growth rates of consumption downwards (upwards) by a relatively large percentage after the occurrence of a negative (positive) stock return shock. Figure 1 illustrates the sensitivity q_h as a function of the horizon h for various parameter values.

Denote by $\Sigma_{t,h}$ the annualized volatility of future consumption $\log c^*_{t+h}$ at time t, i.e.,

$$\Sigma_{t,h} := \sqrt{\frac{\mathbb{V}_t\left[\log c^*_{t+h}\right]}{h}}. \tag{27}$$

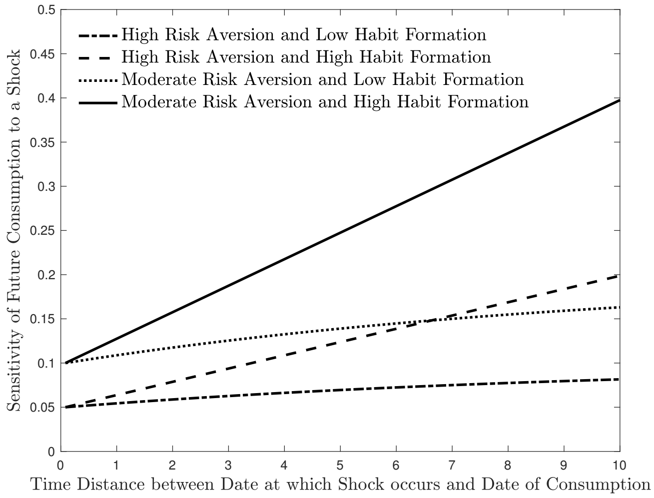


Figure 1: **Sensitivity of future consumption.** The figure illustrates the sensitivity of future consumption to a stock return shock (i.e., q_h) as a function of the horizon h (i.e., the time distance between the date at which the stock return shock occurs and the date of consumption). The figure considers four different types of individuals: a highly risk-averse individual with a low degree of habit persistence (i.e., $\gamma = 20$, $\alpha = 0.2$, $\beta = 0.1$); a highly risk-averse individual with a high degree of habit persistence (i.e., $\gamma = 20$, $\alpha = \beta = 0.3$); a moderately risk-averse individual with a low degree of habit persistence (i.e., $\gamma = 10$, $\alpha = 0.2$, $\beta = 0.1$); and a moderately risk-averse individual with a high degree of habit persistence (i.e., $\gamma = 10$, $\alpha = \beta = 0.3$). In the case of a CRRA individual, the sensitivity of future consumption does not depend on the horizon h. We set both the market price of risk λ and the stock return volatility σ equal to 0.2.

Here, $\mathbb{V}_t$ denotes the variance conditional on the information available at time t. We find that the annualized volatility of the future consumption choice of an individual whose preferences exhibit internal habit formation depends on the horizon h. More specifically, the annualized volatility of $\log c^*_{t+h}$ is given in closed-form by

$$\Sigma_{t,h} = \Sigma_h = \sqrt{\frac{\int_0^h q_v^2 \, \mathrm{d}v}{h}} \cdot \sigma. \tag{28}$$

Note that the annualized variance is proportional to the normalized integrated squared sensitivity q_h. Because q_h increases with the horizon h, it follows that for an individual with habit preferences, the annualized volatility of consumption in the near future is smaller than the annualized volatility of consumption in the far future. Finally, we note that the annualized volatility of the future consumption choice of a CRRA individual does not depend on the horizon h: consumption in the near future exhibits the same annualized volatility as consumption in the far future.

4.3 Shock Absorbing Mechanism

This section illustrates in more detail how the current and future consumption levels of an individual whose preferences exhibit internal habit formation respond to an unexpected stock return shock. We consider an individual who starts working at the age of 25 and passes away at the age of 85. He invests and spends his accumulated wealth according to the ratio internal habit model (23) with preference parameters $\gamma = 10$, $\alpha = 0.3$ and $\beta = 0.3$. We also study our model findings for other degrees of habit persistence.[29] As we show below, our results remain qualitatively unchanged if we vary the degree of habit persistence. We assume that the individual adjusts consumption once a year.[30]

We compare our findings to the optimal behavior of a CRRA individual. We assume that the CRRA individual invests 50% of his accumulated wealth in the stock market.[31] His investment behavior roughly coincides with the investment behavior of a 58-year-old individual with habit preferences; for more details on the portfolio strategy of an individual with habit preferences, see Section 4.5.

Figures 2(a) and (b) illustrate the impact of a 38%[32] stock price *decline* in year one on current and future consumption choices. A CRRA individual fully translates a current stock return shock into his current consumption level. In this illustration, the current consumption level of a CRRA individual decreases by 19.35% after the stock price shock has been realized. The stock return shock does not affect the future growth rates of his consumption; see Figure 2(a) which shows that the shape of the median consumption path of a CRRA individual remains unaffected by a stock return shock.

An individual whose preferences exhibit internal habit formation does not fully translate a current stock return shock into his current consumption level. As a result, the relative decline in the current consumption level of an individual with habit preferences is typically smaller than the relative decline in the current consumption level of a CRRA individual. Indeed, his current consumption level drops by only 4.21% while the current consumption level of a CRRA individual drops by more than 19%. The flip side of protecting current consumption is that the shape of the median

[29] We note that the degrees of habit persistence we explore are considered reasonable by Fuhrer (2000) and Gomes and Michaelides (2003).

[30] All figures and tables in this paper assume that the individual adjusts consumption only once a year. We note that this is *not* a restriction of our framework. We could also illustrate the case in which the individual adjusts consumption every month or every week.

[31] Assuming $\lambda = \sigma = 0.2$, a portfolio weight of 50% implies a relative risk aversion coefficient of 2 in the Merton model (Merton (1969)).

[32] This number corresponds to the decline in the S&P 500 index between January 1, 2008 and December 31, 2008.

consumption path cannot remain unchanged following a stock return shock; see Figure 2(b) which shows that the individual adjusts the future growth rates of his median consumption downwards. A consequence of adjusting future growth rates is thus that the impact of a shock on median consumption is larger the longer the horizon is.

Figures 2(c) and (d) illustrate the impact of a 24%[33] stock price *increase* in year two on current and future consumption choices. As in Figure 2(a), the CRRA individual directly absorbs the current stock return shock into his current consumption level. The current stock return shock has a smaller impact on the current consumption level of an individual with habit preferences than on that of the CRRA individual. Indeed, an individual with habit preferences has a strong preference to protect current consumption. In fact, in this illustration, he only consumes slightly more than last year, because he has translated part of last year's (negative) stock return shock into consumption of this year. Furthermore, as a result of the current stock price increase, he adjusts the future growth rates of his median consumption upwards; see Figure 2(d).

4.4 Decomposition of the Consumption Dynamics

We can decompose the dynamics of the individual's log consumption choice $\log c_t^*$ as follows (see Appendix A):[34]

$$\mathrm{d}\log c_t^* = g_t\,\mathrm{d}t + p_t\,\mathrm{d}t + \frac{\lambda}{\gamma\sigma}\sigma\,\mathrm{d}W_t. \tag{29}$$

Here,

$$g_t = \frac{1}{\gamma}\left(\widehat{r}_t + \frac{1}{2}\lambda^2 - \delta\right), \tag{30}$$

$$p_t = Q_t'\log c_0^* + \frac{1}{\gamma}\int_0^t Q_{t-s}' g_s\,\mathrm{d}s + \frac{\lambda}{\gamma\sigma}\int_0^t Q_{t-s}'\sigma\,\mathrm{d}W_s, \tag{31}$$

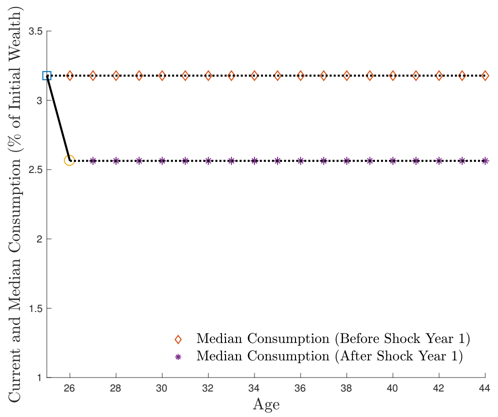
with $\widehat{r}_t = \beta + (r - \alpha\beta P_t)\,/\,(1+\beta P_t)$, $Q_{t-s}' = \mathrm{d}Q_{t-s}/\,\mathrm{d}t$, and P_t and Q_{t-s} defined in (20) and (26), respectively.

The right-hand side of Eqn. (29) consists of three terms. The first term $g_t\,\mathrm{d}t$ represents the unconditional median growth rate of log consumption. Two counteracting forces

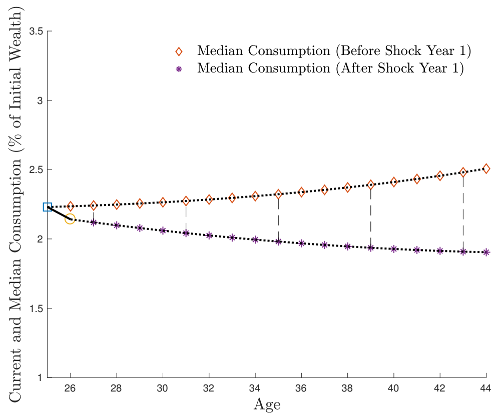
[33]This number corresponds to the increase in the S&P 500 index between January 1, 2009 and December 31, 2009.

[34]Appendix B studies the optimal consumption dynamics in the case the terminal time T equals the individual's uncertain date of death.

-38% stock return shock in year 1

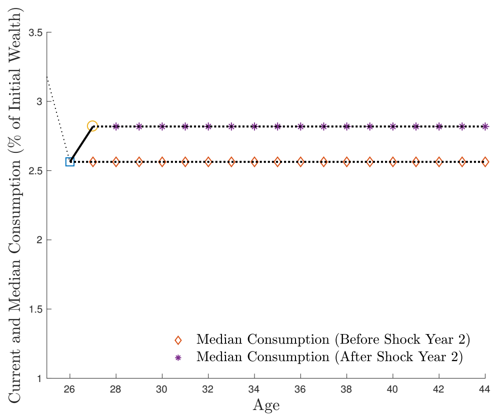


(a) CRRA Individual

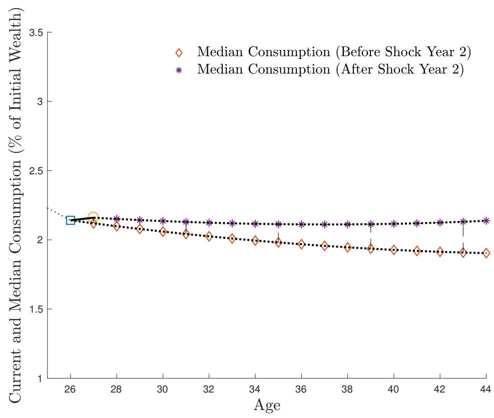


(b) Individual with Habit Preferences

$+24\%$ stock return shock in year 2 (next to the -38% stock return shock in year 1)



(c) CRRA Individual



(d) Individual with Habit Preferences

Figure 2: **Shock absorbing mechanisms.** The figure shows the impact of stock return shocks on current and future consumption choices. The left panels consider a CRRA individual, while the right panels consider an individual whose preferences exhibit internal habit formation (with preference parameters $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$). The small solid lines represent the change in current consumption as a result of the shock. The CRRA individual invests 50% of his accumulated wealth in the stock market (i.e., his relative risk aversion coefficient is equal to 2). Wealth at the age of 25 is for both individuals equal to 45. We set the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%.

determine how large the unconditional median growth rate is. First, the individual has a preference to reduce current consumption (i.e., to increase the unconditional median growth rate of log consumption). Indeed, a decrease in current consumption dampens future habit levels. Furthermore, it increases expected investment earnings, because the individual will be able to save more. The values for the parameters α, β, r and λ jointly determine the strength of the first force. Second, the individual has a preference to increase current consumption (i.e., to reduce the unconditional median growth rate of log consumption). Indeed, the individual is impatient: he prefers to consume sooner rather than later. The value for the preference parameter δ determines the strength of the second force. A large value for δ implies a relatively impatient individual. The second term $p_t \,\mathrm{d}t$ represents past stock return shocks that the individual translates into the current median growth rate of log consumption. This term disappears if preferences do not exhibit internal habit formation (i.e., $\beta = 0$, so that $Q_h = 1$ for all h). Finally, the last term $\lambda/(\gamma\sigma)\cdot\sigma \,\mathrm{d}W_t$ corresponds to the current stock return shock that the individual directly translates into his current consumption level.

Figure 3 illustrates a consumption path for different types of individuals. As shown by this figure, the consumption stream of an individual with habit preferences is smoother than the consumption stream of a CRRA individual. As is well-known, an excessively smooth consumption stream is also consistent with aggregate consumption data (see, e.g., Flavin (1985), Deaton (1987), and Campbell and Deaton (1989)) and other behavioral models (see, e.g., Kőszegi and Rabin (2006, 2007, 2009), Pagel (2017), and Van Bilsen et al. (2017)).

4.5 Optimal Portfolio Choice

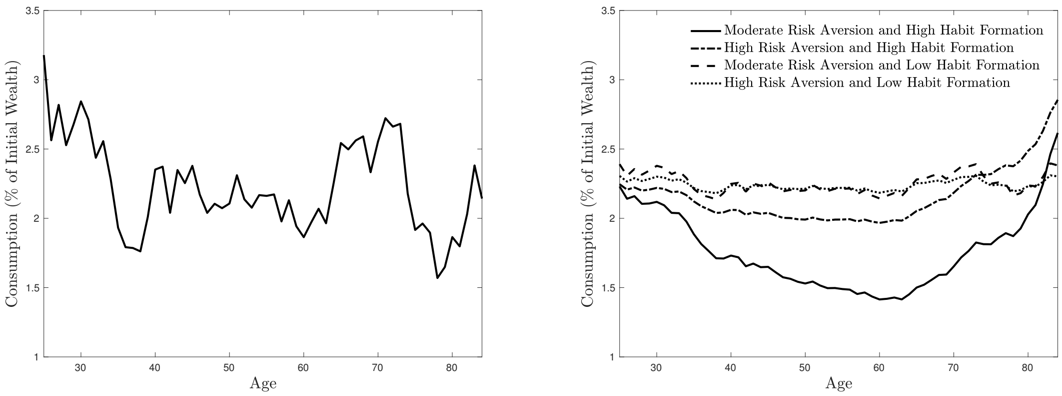
Theorem 4.2 presents the (approximate) optimal portfolio choice π_t^*.

Theorem 4.2. *Consider an individual with lifetime utility* (23) *and habit formation process* (8) *who solves the consumption and portfolio choice problem* (18). *Then the (approximate) optimal portfolio choice π_t^* is given by*

$$\pi_t^* = \int_0^{T-t} q_h \frac{V_{t,h}}{V_t} \,\mathrm{d}h \cdot A_t. \tag{32}$$

Here, $V_t = \int_0^{T-t} V_{t,h} \,\mathrm{d}h$ and $V_{t,h}$ denotes the market value at time t of c_{t+h}^*. Appendix A provides an explicit analytical expression for $V_{t,h}$ (see (66)*).*

Figure 4 illustrates the portfolio strategy π_t^*/A_t of an individual with habit preferences. The individual implements a life-cycle investment strategy: the share of accumulated wealth invested in the risky stock decreases as the individual ages. Indeed, the individual has less time to absorb a stock return shock as he grows older. We observe that the larger



(a) CRRA Individual

(b) Habit Preferences

Figure 3: **Consumption dynamics.** Panel A illustrates a consumption path of a CRRA individual, while panel B shows the impact of internal habit formation on the consumption dynamics. Panel B considers four different types of individuals: a moderately risk-averse individual with a high degree of habit persistence (i.e., $\gamma = 10$, $\alpha = \beta = 0.3$), a highly risk-averse individual with a high degree of habit persistence (i.e., $\gamma = 20$, $\alpha = \beta = 0.3$); a moderately risk-averse individual with a low degree of habit persistence (i.e., $\gamma = 10$, $\alpha = 0.2$, $\beta = 0.1$), and a highly risk-averse individual with a low degree of habit persistence (i.e., $\gamma = 20$, $\alpha = 0.2$, $\beta = 0.1$). The CRRA individual invests 50% of his accumulated wealth in the stock market (i.e., his relative risk aversion coefficient is equal to 2). Wealth at the age of 25 is for every individual equal to 45. We set the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%. Individuals adjust consumption once a year.

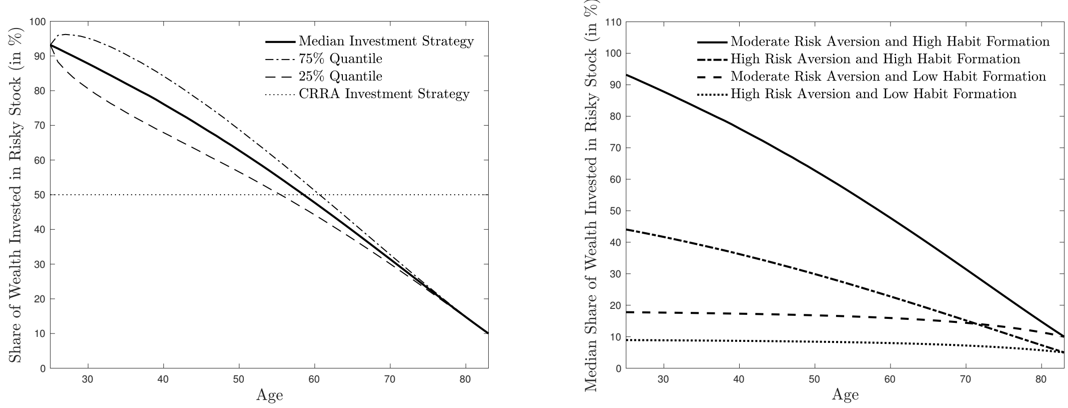
the degree of habit persistence, the more pronounced the life-cycle investment strategy will be; see Figure 4(b). A declining equity glide path during both the accumulation and the retirement phase is also commonly adopted by target date fund managers; see Morningstar (2017). The portfolio strategy of an individual with habit preferences stands in sharp contrast to the portfolio strategy of a CRRA individual. Such an individual implements an age-independent portfolio strategy; see the dotted line in Figure 4(a).[35]

Figure 4(a) also shows that the portfolio strategy of an individual with habit preferences hardly varies with the state of the economy, especially at higher ages.[36] The

[35] We note that a CRRA individual invests a constant share of *total* wealth, which equals the sum of financial wealth and human capital, in the risky stock.

[36] A state-independent portfolio strategy has three key advantages for annuity providers. First, an annuity provider can implement the portfolio strategy without much effort: he does not have to monitor any state variables. Second, an annuity with a state-independent portfolio strategy is easy to communicate as the equity glide path is known at inception. Third, the individual typically achieves a prosperous expected payout stream at an affordable price. Indeed, if an annuity provider offers an annuity with a state-dependent portfolio strategy, then this portfolio strategy is often designed such that it protects customers against losses or locks in investment gains. While attractive from the viewpoint of

portfolio strategy is not completely state-independent: while the sensitivity q_h and volatility Σ_h of future consumption are fully state-independent due to the constant relative risk aversion property, a shock to the economy alters the shape of the median consumption stream (see Figure 2). In particular, long horizons benefit relatively more from a positive shock, while, on the other hand, short horizons suffer relatively less from a negative shock. As a result, the value weights $V_{t,h}/V_t$ in (32) change following a shock. However, this effect is small (second-order), so that the portfolio strategy is nearly insensitive to economic shocks.



(a) Summary Statistics

(b) Comparative Statics

Figure 4: **Investment strategy.** Panel A shows summary statistics of the investment strategy of an individual whose preferences exhibit internal habit formation (with preference parameters $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$). Panel B illustrates how internal habit formation affects the median investment strategy. This panel considers four different types of individuals: a moderately risk-averse individual with a high degree of habit persistence (i.e., $\gamma = 10$, $\alpha = \beta = 0.3$), a highly risk-averse individual with a high degree of habit persistence (i.e., $\gamma = 20$, $\alpha = \beta = 0.3$); a moderately risk-averse individual with a low degree of habit persistence (i.e., $\gamma = 10$, $\alpha = 0.2$, $\beta = 0.1$), and a highly risk-averse individual with a low degree of habit persistence (i.e., $\gamma = 20$, $\alpha = 0.2$, $\beta = 0.1$). Wealth at the age of 25 is for every individual equal to 45. We set the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%. Individuals adjusts consumption once a year.

Table 1 shows the (median) year-on-year volatility of accumulated wealth for various ages. The year-on-year consumption volatility is always equal to 2%, irrespective of the individual's current age.[37] With ratio internal habit formation, the year-on-year consumption volatility is smaller than the year-on-year volatility of accumulated wealth.

avoiding losses, the flip side of this investment behavior is that upward potential can be rather limited.

[37]We note that the year-on-year consumption volatility is given by $\lambda/(\gamma\sigma) \cdot \sigma$ (see Eqn. (29)). Hence, assuming $\lambda = \sigma = 0.2$ and $\gamma = 10$, we find that the year-on-year consumption volatility is equal to 2%.

We find that the degree of habit persistence largely determines the share of accumulated wealth invested in the risky stock, while the individual's coefficient of relative risk aversion largely determines the degree of variability of current consumption. As a result, given a certain degree of relative risk aversion, an individual with habit preferences invests more in the stock market early in life than an individual with conventional CRRA preferences. An individual with habit preferences translates a stock return shock not only in current consumption but also in future growth rates of consumption. This enables the individual to take a relatively risky position in the stock market at young ages.

Age	Median Year-on-Year Volatility of Wealth (%)
25	18.63
35	16.42
45	13.93
55	11.09
65	7.94
75	4.60
83	2.00

Table 1: **Median year-on-year volatility of wealth.** The table reports the median year-on-year volatility of wealth for various ages. The year-on-year consumption volatility is always equal to 2%, irrespective of the individual's age. The individual's preference parameters are: $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$. Wealth at the age of 25 is equal to 45. We set the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%. The individual adjusts consumption once a year.

5 Internal Habits and Stochastic Interest Rates

This section explores the implications of a stochastic interest rate for the optimal consumption and portfolio choice of an individual with ratio internal habit preferences. We assume that the economy consists of three assets: one (locally) risk-free asset, a risky stock, and a risky zero-coupon bond with time to maturity T_1. The price of the risk-free asset, B_t, and the (2×1)-vector of risky asset prices, S_t, satisfy[38]

$$\frac{\mathrm{d}B_t}{B_t} = r_t \,\mathrm{d}t, \tag{33}$$

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t, \tag{34}$$

where the risk-free interest rate r_t follows an Ornstein-Uhlenbeck process, i.e.,

$$\mathrm{d}r_t = \kappa \left(\bar{r} - r_t\right) \mathrm{d}t + \begin{bmatrix} \sigma_r \rho \\ \sigma_r \sqrt{1-\rho^2} \end{bmatrix}^\top \mathrm{d}W_t, \tag{35}$$

and μ_t and σ_t are defined as follows:

$$\mu_t = \begin{bmatrix} r_t + \lambda_1 \sigma_S \\ r_t - \sigma_r D_{T_1} \left(\lambda_1 \rho + \lambda_2 \sqrt{1-\rho^2}\right) \end{bmatrix}, \quad \sigma_t = \begin{bmatrix} \sigma_S & 0 \\ -\sigma_r D_{T_1} \rho & -\sigma_r D_{T_1} \sqrt{1-\rho^2} \end{bmatrix}. \tag{36}$$

Here, $\kappa \geq 0$ denotes the mean reversion coefficient, $\bar{r}$ corresponds to the long-term interest rate, $\sigma_r > 0$ stands for the interest rate volatility, $-1 \leq \rho \leq 1$ models the correlation between the interest rate and the risky stock price, $\sigma_S > 0$ represents the stock return volatility, and $D_{T_1} = \frac{1}{\kappa}\left(1 - e^{-\kappa T_1}\right)$ denotes the interest rate sensitivity of the bond. The market prices of risk associated with the two Brownian increments are given by λ_1 and λ_2. Appendix A proves the following theorem.

Theorem 5.1. *Consider an individual with lifetime utility* (23) *and habit formation process* (8) *who solves the consumption and portfolio choice problem* (18)*. Assume that the interest rate r_t satisfies* (35) *and that the economy consists of a (locally) risk-free asset, a stock, and a zero-coupon bond with time to maturity T_1. Let the dynamics of the risky assets be given by* (34)*. Then the optimal amounts of wealth invested in the stock and bond are given by*

$$\pi^*_{1,t} = -\frac{1}{\sigma_S} \frac{\partial V_t}{\partial \log \widehat{M}_t} \frac{1}{V_t} \cdot \left(\widehat{\lambda}_{1,t} - \frac{\rho}{\sqrt{1-\rho^2}} \widehat{\lambda}_{2,t}\right) \cdot A_t, \tag{37}$$

$$\pi^*_{2,t} = \frac{\widehat{\lambda}_{2,t}}{\sigma_r \sqrt{1-\rho^2} D_{T_1}} \cdot \frac{\partial V_t}{\partial \log \widehat{M}_t} \frac{1}{V_t} \cdot A_t - \frac{1}{D_{T_1}} \cdot \frac{\partial V_t}{\partial r_t}, \tag{38}$$

[38]We note that this economy emerges as a special case of the economy considered by Brennan and Xia (2002).

with $V_t = \int_0^{T-t} V_{t,h}\,\mathrm{d}h$ *representing the market value of the future (approximate) optimal consumption stream* $\{c_s^*\}_{t \le s \le T}$ *and*

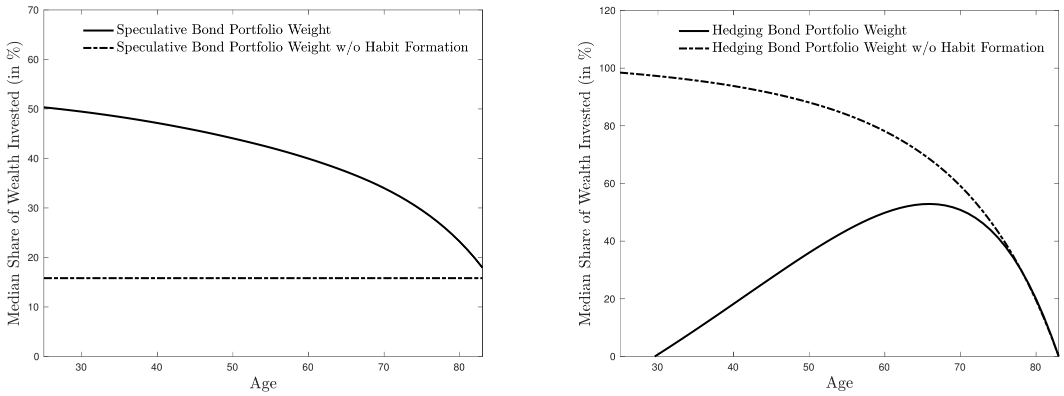
$$\widehat{\lambda}_{1,t} = \lambda_1 + \beta \frac{\sigma_r \rho \widehat{D}_t P_t}{1 + \beta P_t}, \tag{39}$$

$$\widehat{\lambda}_{2,t} = \lambda_2 + \beta \frac{\sigma_r \sqrt{1-\rho^2}\,\widehat{D}_t P_t}{1 + \beta P_t}, \tag{40}$$

with $\widehat{M_t}$ *and* P_t *given by* (19) *and* (20)*, respectively, and* $\widehat{D}_t$ *defined in Appendix A* (see (75)).

Figure 5(a) shows the first component of the bond portfolio weight $\pi_{2,t}^*/A_t$ (see (38)) as a function of age. We call this component the speculative bond portfolio weight. Two counteracting forces determine how this speculative weight evolves over the individual's life cycle. On the one hand, the available time to incorporate a speculative shock into future consumption declines with age. As a result, the speculative demand decreases as the individual becomes older. A similar reasoning applies to the stock portfolio weight; see Section 4.5. On the other hand, the older the individual, the more sensitive the individual's relative consumption choice $\widehat{c}_t^*$ (typically) is to interest rate shocks; see Eqn. (78) in Appendix A which shows that $\widehat{\lambda}_{2,t}/\gamma$ models the interest rate sensitivity of $\widehat{c}_t^*$. Note that $\widehat{\lambda}_{2,t}$ becomes more negative as the individual ages. This causes the speculative demand to increase with age. The first effect dominates the second effect in Figure 5(a).

Figure 5(b) shows the second component of the bond portfolio weight $\pi_{2,t}^*/A_t$ (see again (38)) as a function of age. We call this component the hedging bond portfolio weight. The value of the hedging weight is also the result of two counteracting forces: a horizon effect and a substitution effect. On the one hand, the longer the horizon h, the larger the impact of a shock in the interest rate will be on the price of future consumption. This causes the hedging portfolio weight to decrease over the life cycle. On the other hand, we find a new effect that causes the hedging bond portfolio weight to increase over the life cycle. We can explain this effect by the fact that the willingness of a habit-forming individual to substitute consumption over time decreases with age. Indeed, as the individual ages, the duration of remaining lifetime consumption declines, and hence the current habit level determines to a greater extent future consumption levels. Jointly, these two effects lead to a hump-shaped pattern. Finally, we note that the second effect may explain why not many young individuals include long-terms bonds in their investment portfolios; see Morningstar (2017).



(a) Speculative Bond Portfolio Weights

(b) Hedging Bond Portfolio Weights

Figure 5: **Portfolio choice with stochastic interest rates.** Panel A illustrates the median speculative bond portfolio weight (with and without habit formation) as a function of age. We assume that the individual invests wealth in a zero-coupon bond with a fixed time to maturity of 10 (i.e., $T_1 = 10$). Panel B illustrates the median hedging bond portfolio weight (with and without habit formation) as a function of age. The individual's preference parameters are as follows: $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$ (for the case of no habit formation, we have $\alpha = \beta = 0$). We set the long-term interest rate $\bar{r}$ equal to 1%, the mean reversion parameter κ to 0.1, the interest rate volatility σ_r to 2%, the market price of interest rate risk λ_2 to -0.2, the market price of stock market risk λ_1 to 0.2, the stock return volatility σ_S to 20%, and the subjective rate of time preference δ to 3%. The individual adjusts consumption once a year. Note that part of the individual's wealth is invested in the money market account.

6 Internal Habits and Epstein-Zin Utility

As shown in Appendix C, an individual with habit preferences prefers (unrealistically) high unconditional median growth rates of log consumption (especially at high ages) except when his time preference rate δ is excessive.[39] This section therefore considers a utility specification that disentangles the elasticity of intertemporal substitution from the coefficient of relative risk aversion. Under this extended preference model, quite remarkably, median consumption growth can be low or moderate even when the individual's time preference rate δ takes on reasonable values.

6.1 Utility Specification

We consider an individual with Epstein-Zin utility in terms of relative consumption. Let $\{U_t\}_{0 \le t \le T}$ be the utility process. We assume that $\{U_t\}_{0 \le t \le T}$ satisfies the following

[39] Indeed, as already pointed out by Deaton (1992), an individual with habit preferences derives utility not only from consumption levels but also from consumption growth.

integral equation ($0 \leq t \leq T$):

$$U_t\left(\frac{c}{h}\right) = \mathbb{E}_t\left[\int_t^T f\left(\frac{c_s}{h_s}, U_s\right) \mathrm{d}s\right]. \tag{41}$$

Here, $\mathbb{E}_t$ denotes the expectation conditional upon information at time t. The intertemporal aggregator f is assumed to be given by[40]

$$f\left(\frac{c_t}{h_t}, U_t\right) = (1+\zeta)\left[\frac{\left(\frac{c_t}{h_t}\right)^{\varphi}}{\varphi}|U_t|^{\frac{\zeta}{1+\zeta}} - \delta U_t\right]. \tag{42}$$

Here, $\zeta > -1$ and $\varphi < \min\{1, 1/(1+\zeta)\}$ are preference parameters. We refer to (42) as the Kreps-Porteus aggregator (Kreps and Porteus (1978)).[41] The individual maximizes $U_0\left(\frac{c}{h}\right)$ (see (41)) with $f\left(\frac{c_t}{h_t}, U_t\right)$ given by (42) subject to the habit process (8) and the dynamic budget constraint (7).

6.2 Dynamic Consumption and Portfolio Choice

We can solve the individual's optimization problem by first invoking our pathwise approximation approach and next the approach of Schroder and Skiadas (1999). The following theorem presents the (approximate) optimal consumption choice.

Theorem 6.1. *Consider an individual with utility process* (41)*, intertemporal aggregator* (42) *and habit formation process* (8) *who solves the consumption and portfolio choice problem* (18)*. Assume constant investment opportunities (i.e., $r_t = r$, $\mu_t = \mu$, $\sigma_t = \sigma$ and $\lambda_t = \lambda$ for all t). Let h_t^* be the individual's habit level implied by substituting the individual's optimal past relative consumption choices $\widehat{c}_s^*$ ($s \leq t$) into* (15) *and let z be a scaling parameter associated with the static budget constraint in* (18)*. Then the*

[40] If $\varphi = 0$, then (42) reduces to $f(c_t/h_t, U_t) = (1+\zeta U_t)[\log\{c_t/h_t\} - (\delta/\zeta)\log\{1+\zeta U_t\}]$.

[41] If $\zeta = 0$ and the habit level h_t equals unity (i.e., $\alpha = \beta = 0$), then $f(c_t/h_t, U_t)$ reduces to

$$f\left(\frac{c_t}{h_t}, U_t\right) = \frac{1}{\varphi}c_t^{\varphi} - \delta U_t. \tag{43}$$

Eqn. (41) is then equivalent to the additive utility specification

$$U_t\left(\frac{c}{h}\right) = \mathbb{E}_t\left[\int_t^T e^{-\delta(s-t)}\frac{1}{\varphi}c_s^{\varphi}\mathrm{d}s\right]. \tag{44}$$

individual's (approximate) optimal consumption choice c_t^* is given by

$$c_t^* = h_t^* z \exp\left\{\int_0^t \left(\psi\left[\widehat{r}_s + \frac{1}{2}\frac{\lambda^2}{\gamma} - \delta\right] + \frac{1}{2}\frac{\lambda^2(\gamma-1)}{\gamma^2}\right)\mathrm{d}s + \frac{\lambda}{\gamma}\int_0^t \mathrm{d}W_s\right\}, \tag{45}$$

where $\psi = 1/(1-\varphi)$ and $\gamma = 1 - \varphi(1+\zeta)$. The scaling parameter $z \geq 0$ is determined such that the individual's original budget constraint holds with equality.

From (45) one may verify that the sensitivity q_h and volatility Σ_h of future consumption take the same form as in the base-line model (see Section 4). In the preference model of this section, the parameter ψ models the individual's willingness to substitute consumption over time. Relative risk aversion is thus decoupled from the elasticity of intertemporal substitution. Figure 6 illustrates the median consumption path as a function of age for an individual whose preferences combine Epstein-Zin utility with the ratio internal habit model. As in Section 4, we assume $\alpha = \beta = 0.3$ and $\gamma = 10$. Figure 6 shows that the growth rates of the individual's median consumption path are substantially lower at high ages compared to the case without Epstein-Zin utility. Indeed, if the elasticity of intertemporal substitution is relatively low (as is the case in Figure 6 where ψ equals zero), the individual is less willing to substitute current consumption for future consumption in order to avoid large future habit levels.

The general expression for the (approximate) optimal portfolio choice under Epstein-Zin utility in an economy with one risky asset remains the same as in Section 4; see, in particular, Eqn. (32). However, under Epstein-Zin utility, long horizons receive smaller value weights in the computation of the portfolio strategy compared to the case without Epstein-Zin utility, as wealth accumulation during retirement is not excessive. As a result, an individual whose preferences combine Epstein-Zin utility with habit formation invests less in the risky stock than an individual whose preferences are described by the ratio internal habit model without Epstein-Zin utility; see Figure 7 which shows the reduction in the share of wealth invested in the risky stock as a result of superimposing Epstein-Zin utility to our base-line model.

7 Accuracy of the Approximation Method

The consumption and portfolio strategies presented in Sections 4, 5 and 6 are exact only in the case when $\beta = 0$ and/or $\alpha = \infty$. In all other cases, the consumption and portfolio strategies are approximate based upon linearizing the individual's static budget

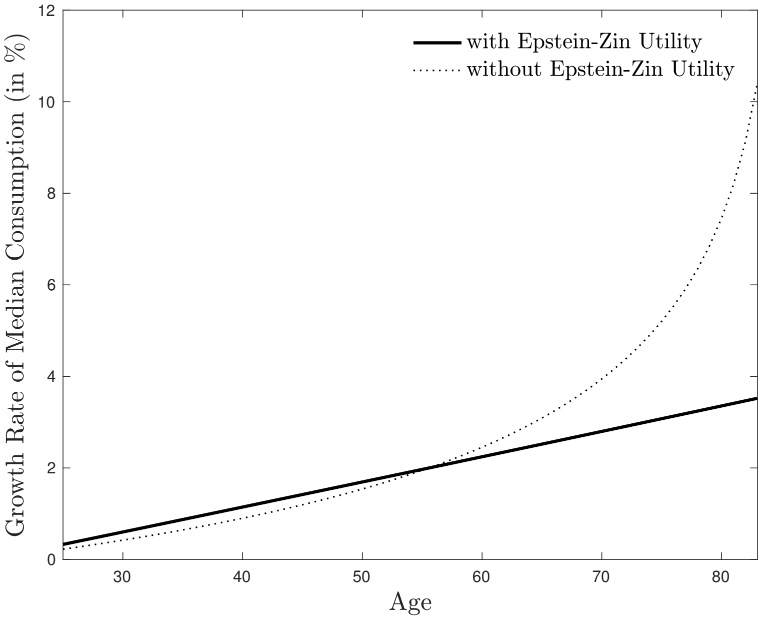


Figure 6: **Median consumption path.** The figure illustrates the median consumption path as a function of age for an individual whose preferences combine Epstein-Zin utility with the ratio internal habit model. The preference parameters are: $\psi = 0$, $\gamma = 10$, $\alpha = 0.3$ and $\beta = 0.3$. Wealth at the age of 25 is equal to 45. For comparison purposes, we also plot the median consumption path for the case without Epstein-Zin utility; see the dotted line. We set the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%. The individual adjusts consumption once a year.

constraint in (16) around the relative consumption trajectory $\{\widehat{c}_t\}_{0\leq t\leq T} = x$ $(x > 0)$. This section analyzes the approximation error induced by applying a pathwise linearization to the static budget constraint.

We consider an individual whose preferences are represented by (41) with aggregator (42) and habit formation process (8). We determine the genuine optimal consumption choice c_t^{opt} and optimal portfolio choice π_t^{opt} by using the method of backward induction; Appendix D provides details on the numerical solution technique. We evaluate the performance of the approximate optimal consumption choice c_t^* by measuring the relative decline in certainty equivalent consumption.[42] Table 2 reports our results. We find that the approximation error is a decreasing function of γ, and an

[42]The certainty equivalent of an uncertain consumption strategy is defined to be the constant consumption level that yields indifference to the uncertain consumption strategy. The certainty equivalent consumption choice ce always exists if $\alpha \geq \beta$. In particular, lifetime utility $U(c/h)$ is increasing in certainty equivalent consumption ce if $\beta \int_0^T e^{-\alpha t}\,\mathrm{d}t \leq 1$. If T is large, then $\int_0^T e^{-\alpha t}\,\mathrm{d}t \approx \frac{1}{\alpha}$. Hence, we can always compute (for any T) the certainty equivalent consumption choice ce if $\frac{\beta}{\alpha} \leq 1$.

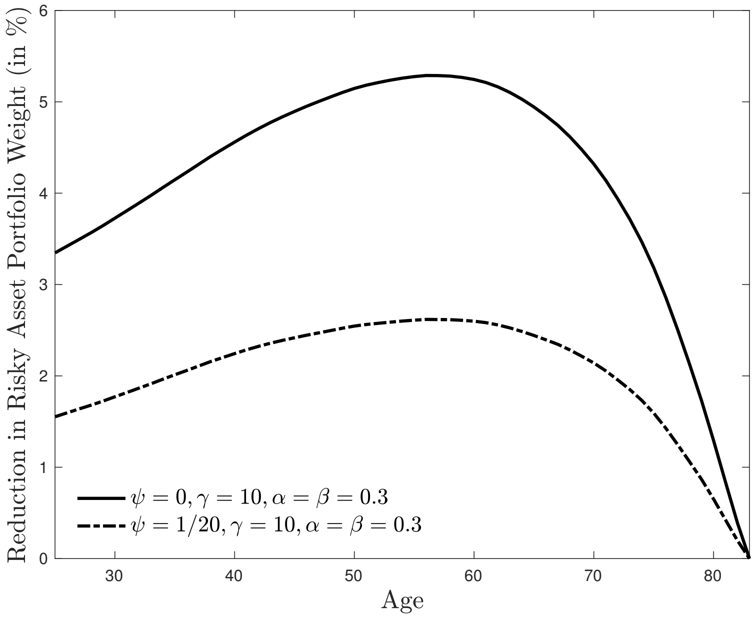


Figure 7: **Reduction in risky stock portfolio weight.** The figure illustrates the reduction in the share of wealth invested in the risky stock (in %) as a result of superimposing Epstein-Zin utility to our base-line model as a function of age. The preference parameters are: $\psi = 0$, $\gamma = 10$, $\alpha = \beta = 0.3$ (solid line) and $\psi = 1/20$, $\gamma = 10$, $\alpha = \beta = 0.3$ (dash-dotted line). Wealth at the age of 25 is equal to 45. We set the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%. The individual adjusts consumption once a year.

increasing function of β. Indeed, if γ is large, the habit level closely tracks consumption. Also, if β is small, habit formation is rather limited. In nearly all cases, the approximation error is smaller than 1%. Furthermore, we note that Table 2 only considers cases for which α equals β. If β is smaller than α, the welfare loss will be lower. In particular, in the limiting case $\beta = 0$, the welfare loss will vanish. For illustration purposes, Figure 8 also compares, for three different economic scenarios, the optimal consumption path with the approximate consumption path. We observe a close match. To assess the accuracy of the approximation method further, Figure 9 shows $c_t^{\text{opt}}/h_t^{\text{opt}}$ for various sets of parameter values. We note that if $c_t^{\text{opt}}/h_t^{\text{opt}}$ is close to one, the approximation error is small. We find that the histograms are centered around one and that, as expected, the histogram width decreases when γ goes up.

Finally, we compute the minimum welfare loss associated with implementing the Merton consumption strategy (Merton (1969)). This consumption strategy is characterized by the degree of relative risk aversion of the Merton individual. We

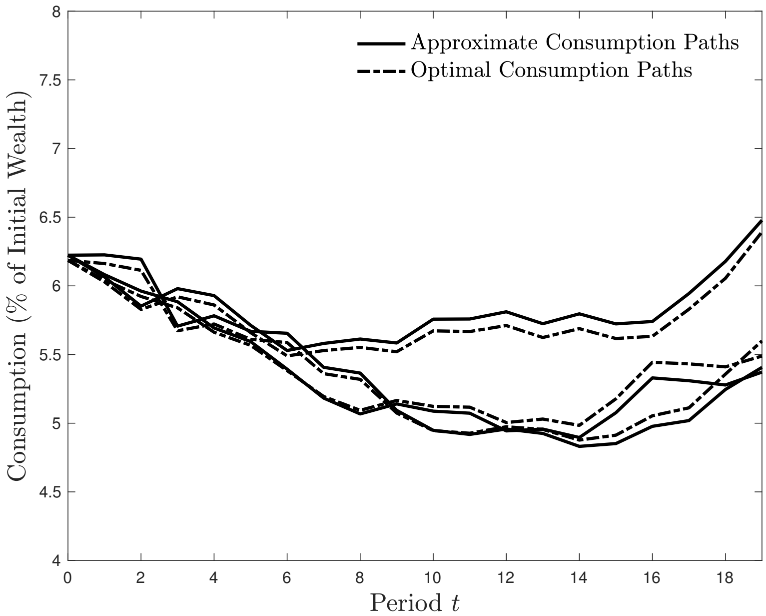


Figure 8: **Consumption trajectories.** The figure compares, for three different economic scenarios, the optimal consumption path with the approximate consumption path. The preference parameters are: $\psi = 1/10$, $\gamma = 10$, $\alpha = 0.3$ and $\beta = 0.3$. Initial wealth equals 15. We set the terminal time T equal to 20, the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%. The individual adjusts consumption once a year.

assume that the habit-forming individual is restricted to implement the Merton consumption strategy. He chooses the relative risk aversion coefficient in the Merton model such that the difference between his utility associated with implementing the Merton consumption strategy and his utility associated with implementing the approximate optimal consumption strategy is minimal. Table 3 reports our results for various sets of parameter values. We find that the minimum welfare loss due to the Merton consumption strategy is likely a factor 10 larger than the welfare loss associated with our approximation method.

8 Concluding Remarks

This paper has explored how an individual who derives utility from the ratio between his consumption and an endogenous habit should consume and invest over the life cycle. It is well-known that analytical closed-form solutions to multiplicative internal habit

γ	ψ	α	β	δ	A_0	Welfare Loss (%)
6	1/6	0.30	0.30	0.03	15	1.0012
8	1/8	0.30	0.30	0.03	15	0.3967
10	1/10	**0.30**	**0.30**	**0.03**	**15**	**0.1926**
12	1/12	0.30	0.30	0.03	15	0.1006
14	1/14	0.30	0.30	0.03	15	0.0642

(a) Sensitivity with respect to the Relative Risk Aversion Coefficient γ

γ	ψ	α	β	δ	A_0	Welfare Loss (%)
10	1/10	0.20	0.20	0.03	15	0.0764
10	1/10	0.25	0.25	0.03	15	0.1237
10	1/10	**0.30**	**0.30**	**0.03**	**15**	**0.1926**
10	1/10	0.35	0.35	0.03	15	0.2827
10	1/10	0.40	0.40	0.03	15	0.3826

(b) Sensitivity with respect to the Degree of Habit Formation $\alpha = \beta$

γ	ψ	α	β	δ	A_0	Welfare Loss (%)
10	1/10	0.30	0.30	0.01	15	0.2090
10	1/10	0.30	0.30	0.02	15	0.2037
10	**1/10**	**0.30**	**0.30**	**0.03**	**15**	**0.1926**
10	1/10	0.30	0.30	0.04	15	0.1897
10	1/10	0.30	0.30	0.05	15	0.1783

(c) Sensitivity with respect to the Time Discount Rate δ

γ	ψ	α	β	δ	A_0	Welfare Loss (%)
10	1/10	0.30	0.30	0.03	13	0.1927
10	1/10	0.30	0.30	0.03	14	0.1881
10	1/10	**0.30**	**0.30**	**0.03**	**15**	**0.1926**
10	1/10	0.30	0.30	0.03	16	0.2081
10	1/10	0.30	0.30	0.03	17	0.2610

(d) Sensitivity with respect to the Initial Wealth Level A_0

γ	ψ	α	β	δ	A_0	Welfare Loss (%)
10	1/14	0.30	0.30	0.03	15	0.2773
10	1/12	0.30	0.30	0.03	15	0.2506
10	1/10	**0.30**	**0.30**	**0.03**	**15**	**0.1926**
10	1/8	0.30	0.30	0.03	15	0.4286
10	1/6	0.30	0.30	0.03	15	1.2682

(e) Sensitivity with respect to the Preference Parameter ψ

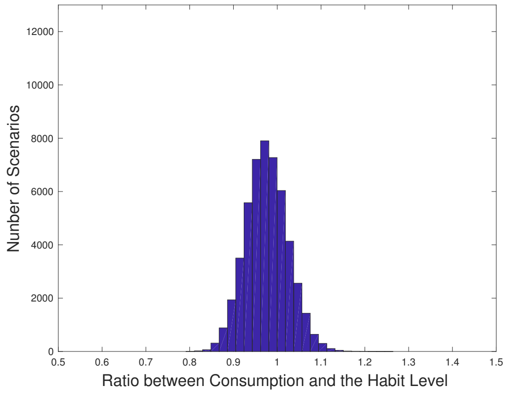
Table 2: **Welfare losses.** The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the approximate optimal consumption choice (24). We set the terminal time T equal to 20, the risk-free interest rate r to 1%, the market price of risk λ to 0.2, and the stock return volatility σ to 20%. The individual adjusts consumption once a year.

γ	ψ	α	β	δ	A_0	Welfare Loss (%)
6	1/6	0.30	0.30	0.03	15	2.8751
14	1/14	0.30	0.30	0.03	15	2.5241
10	1/10	**0.30**	**0.30**	**0.03**	**15**	**2.3061**
10	1/10	0.20	0.20	0.03	15	1.3031
10	1/10	0.40	0.40	0.03	15	3.3414

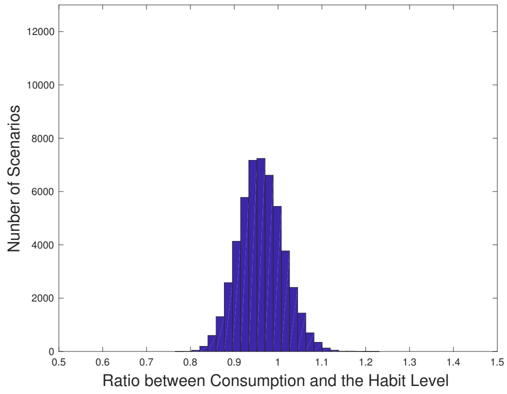
Table 3: **Minimum welfare losses.** The table reports the minimum welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the Merton consumption strategy. We set the terminal time T equal to 20, the risk-free interest rate r to 1%, the market price of risk λ to 0.2, and the stock return volatility σ to 20%. The individual adjusts consumption once a year.

models do not exist in general. Therefore, we have developed a general solution procedure based on a linearization of the static budget constraint around the endogenous habit level enabling us to transform consumption and portfolio problems with multiplicative internal habits into approximate consumption and portfolio problems without habits.

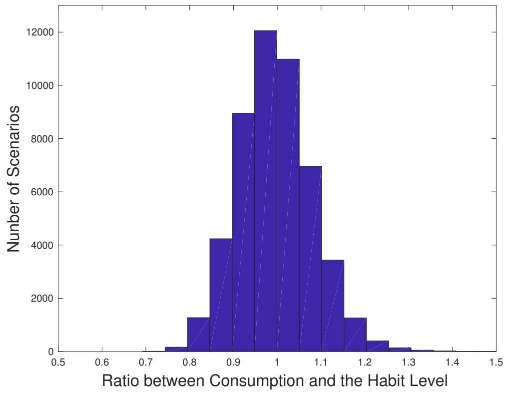
We have applied our general solution procedure to three important cases of multiplicative habit formation. The first case considers a constant investment opportunity set and assumes that the individual has additive preferences in terms of relative consumption; see Section 4. We have shown that the individual's preferences induce clearly interpretable implications: the coefficient of relative risk aversion controls the year-on-year volatility of current consumption and the strength of habit persistence controls the extent to which a stock return shock impacts future growth rates of consumption. The second case is an extension that allows for stochastic interest rates and stock-bond investments; see Section 5. We have shown that the speculative bond portfolio weight typically declines with age and that the hedging bond portfolio weight displays a hump-shaped pattern over the life cycle. Finally, we have studied an individual whose preferences combine ratio internal habit formation with Epstein-Zin utility; see Section 6. Interestingly, median consumption now no longer grows at unrealistically high rates at high ages and risky assets become less attractive.



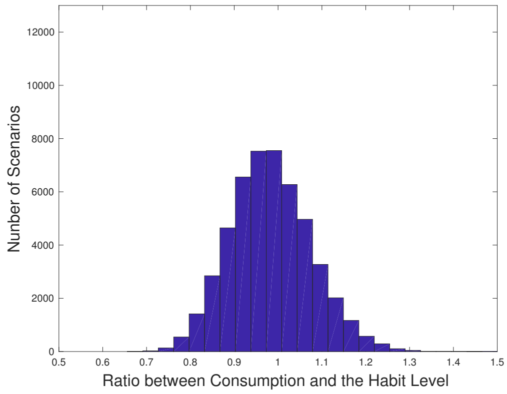
(a) $\gamma = 10, \alpha = \beta = 0.3$



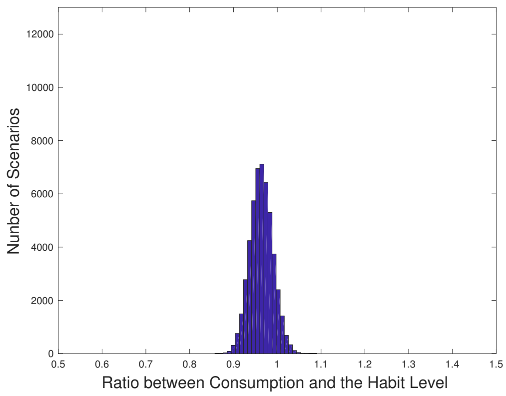
(b) $\gamma = 10, \alpha = \beta = 0.2$



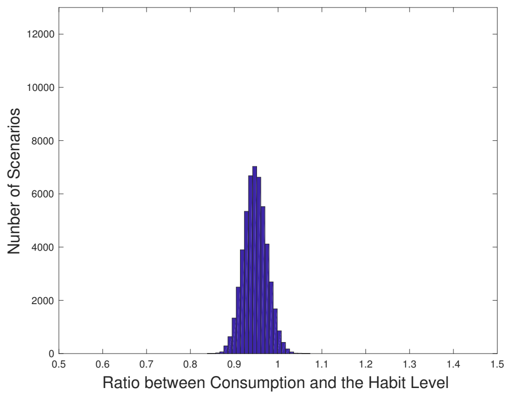
(c) $\gamma = 5, \alpha = \beta = 0.3$



(d) $\gamma = 5, \alpha = \beta = 0.2$



(e) $\gamma = 20, \alpha = \beta = 0.3$



(f) $\gamma = 20, \alpha = \beta = 0.2$

Figure 9: **Accuracy of the pathwise approximation method.** The figure illustrates $c_t^{\text{opt}}/h_t^{\text{opt}}$ for various sets of parameter values. We assume $t = 10$. We set the terminal time T equal to 20, the individual's initial wealth to 15, the risk-free interest rate r to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%.

A Proofs

A.1 Proof of Theorem 3.1

This appendix discusses how to approximate the left-hand side of the new static budget constraint in (16) around the constant consumption trajectory $\{\widehat{c}_t\}_{0\leq t\leq T} = x$ for some positive x. (In the main text we make the simplifying assumption that $\{\widehat{c}_t\}_{0\leq t\leq T} = 1$.) The partial derivative of $\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t$ with respect to the current relative consumption choice $\widehat{c}_t$ is given by

$$\frac{\partial\left(\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right)}{\partial \widehat{c}_t} = M_t h_t \,\mathrm{d}t + \int_t^T M_s \frac{\partial h_s}{\partial \widehat{c}_t} \widehat{c}_s \,\mathrm{d}s. \tag{46}$$

The partial derivative of the future habit level h_s ($s \geq t$) with respect to the current relative consumption choice $\widehat{c}_t$ is given by (this equation follows from differentiating (15) with respect to $\widehat{c}_t$)

$$\frac{\partial h_s}{\partial \widehat{c}_t} = \beta \exp\left\{-(\alpha-\beta)(s-t)\right\} \frac{h_s}{\widehat{c}_t} \,\mathrm{d}t. \tag{47}$$

Substituting (47) into (46) and evaluating (46) around the constant consumption trajectory $\{\widehat{c}_t\}_{0\leq t\leq T} = x$, we arrive at

$$\left.\frac{\partial\left(\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right)}{\partial \widehat{c}_t}\right|_{\{\widehat{c}_t\}_{0\leq t\leq T}=x} = M_t x^{Q_t-1} \,\mathrm{d}t + \beta\left(\int_t^T M_s x^{Q_s-1} e^{-(\alpha-\beta)(s-t)} \,\mathrm{d}s\right) \mathrm{d}t. \tag{48}$$

Here, we define

$$Q_t \coloneqq 1 + \frac{\beta}{\alpha-\beta}\left[1-\exp\left\{-(\alpha-\beta)t\right\}\right]. \tag{49}$$

By virtue of Taylor series expansion up to the first order, we have

$$\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t \approx \int_0^T M_t x^{Q_t} \,\mathrm{d}t + \int_0^T \left.\frac{\partial\left(\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right)}{\partial \widehat{c}_t}\right|_{\{\widehat{c}_t\}_{0\leq t\leq T}=x} \left(\widehat{c}_t - x\right). \tag{50}$$

Substituting (48) into (50), we arrive at

$$\begin{aligned}\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t \approx \int_0^T M_t x^{Q_t} \,\mathrm{d}t + \int_0^T \Bigg[M_t x^{Q_t - 1} \\ + \beta \left(\int_t^T M_s x^{Q_s - 1} e^{-(\alpha-\beta)(s-t)} \,\mathrm{d}s \right) \Bigg] (\widehat{c}_t - x) \,\mathrm{d}t.\end{aligned} \tag{51}$$

Hence, we can approximate the left-hand side of the new static budget constraint in (16) by

$$\begin{aligned}\mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right] &\approx \mathbb{E}\Bigg[\int_0^T M_t x^{Q_t} \,\mathrm{d}t + \int_0^T \Bigg[M_t x^{Q_t-1} \\ &\quad + \beta \left(\int_t^T M_s x^{Q_s-1} e^{-(\alpha-\beta)(s-t)} \,\mathrm{d}s \right) \Bigg] (\widehat{c}_t - x) \,\mathrm{d}t \Bigg] \\ &= \mathbb{E}\Bigg[\int_0^T M_t x^{Q_t} \,\mathrm{d}t + \int_0^T \mathbb{E}_t \Bigg\{ \Bigg[M_t x^{Q_t-1} \\ &\quad + \beta \left(\int_t^T M_s x^{Q_s-1} e^{-(\alpha-\beta)(s-t)} \,\mathrm{d}s \right) \Bigg] (\widehat{c}_t - x) \Bigg\} \,\mathrm{d}t \Bigg] \\ &= \mathbb{E}\Bigg[\int_0^T M_t x^{Q_t} \,\mathrm{d}t + \int_0^T M_t x^{Q_t-1} \mathbb{E}_t \Bigg\{ \Bigg[1 \\ &\quad + \beta \left(\int_t^T \frac{M_s}{M_t} x^{Q_s - Q_t} e^{-(\alpha-\beta)(s-t)} \,\mathrm{d}s \right) \Bigg] (\widehat{c}_t - x) \Bigg\} \,\mathrm{d}t \Bigg] \\ &= \mathbb{E}\left[\int_0^T M_t x^{Q_t} \,\mathrm{d}t + \int_0^T M_t x^{Q_t-1} (1 + \beta P_t) (\widehat{c}_t - x) \,\mathrm{d}t \right] \\ &= -\beta \mathbb{E}\left[\int_0^T M_t x^{Q_t} P_t \,\mathrm{d}t \right] + \mathbb{E}\left[\int_0^T M_t x^{Q_t-1} (1 + \beta P_t) \, \widehat{c}_t \,\mathrm{d}t \right].\end{aligned} \tag{52}$$

Here,

$$P_t = \mathbb{E}_t \left[\int_t^T \frac{M_s}{M_t} x^{Q_s - Q_t} e^{-(\alpha-\beta)(s-t)} \,\mathrm{d}s \right]. \tag{53}$$

We can now establish the approximate optimization problem (18) as follows.

1. First, we replace the left-hand side of the new static budget constraint in (16) by (52).

2. Second, we eliminate the constant term $-\beta\mathbb{E}\left[\int_0^T M_t x^{Q_t} P_t \, \mathrm{d}t\right]$ from (52). This term does not play a role in determining the first-order optimality condition.
3. Finally, we redefine initial wealth A_0 to be $\widehat{A}_0$ such that the approximate optimal consumption strategy $\{c_t^*\}_{0\leq t\leq T} = \{h_t^* \widehat{c}_t^*\}_{0\leq t\leq T}$ is budget-feasible. That is,

$$\mathbb{E}\left[\int_0^T M_t h_t^* \widehat{c}_t^* \, \mathrm{d}t\right] = A_0. \tag{54}$$

Straightforward computations show that the initial wealth $\widehat{A}_0$ associated with the approximate problem is then given by

$$\widehat{A}_0 = A_0 + \mathbb{E}\left[\int_0^T \widehat{M_t} \widehat{c}_t^* \, \mathrm{d}t\right] - \mathbb{E}\left[\int_0^T M_t h_t^* \widehat{c}_t^* \, \mathrm{d}t\right]. \tag{55}$$

Here, $\widehat{M_t} = M_t x^{Q_t - 1}(1 + \beta P_t)$. Note that the value of $\widehat{A}_0$ can only be determined after the problem has been solved.

A.2 Proof of Theorem 4.1

Define $\widehat{M_t} = M_t x^{Q_t - 1}(1 + \beta P_t)$. The Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} &= \mathbb{E}\left[\int_0^T e^{-\delta t} \frac{1}{1-\gamma} (\widehat{c}_t)^{1-\gamma} \, \mathrm{d}t\right] + y\left(\widehat{A}_0 - \mathbb{E}\left[\int_0^T \widehat{M_t} \widehat{c}_t \, \mathrm{d}t\right]\right) \\ &= \int_0^T \mathbb{E}\left[e^{-\delta t} \frac{1}{1-\gamma} (\widehat{c}_t)^{1-\gamma} - y\widehat{M_t}\widehat{c}_t\right] \mathrm{d}t + y\widehat{A}_0. \end{aligned} \tag{56}$$

Here, $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. The individual aims to maximize $e^{-\delta t}\frac{1}{1-\gamma}(\widehat{c}_t)^{1-\gamma} - y\widehat{M_t}\widehat{c}_t$. The approximate optimal relative consumption choice $\widehat{c}_t^*$ satisfies the following first-order optimality condition:

$$e^{-\delta t}(\widehat{c}_t^*)^{-\gamma} = y\widehat{M_t}. \tag{57}$$

After solving the first-order optimality condition, we obtain the following maximum:

$$\widehat{c}_t^* = \left(e^{\delta t} y \widehat{M_t}\right)^{-\frac{1}{\gamma}}. \tag{58}$$

Hence (use (17)),

$$c_t^* = h_t^* \left(y e^{\delta t} \widehat{M_t}\right)^{-\frac{1}{\gamma}}. \tag{59}$$

A verification that the optimal solution to the Lagrangian equals the optimal solution to the static problem (see, e.g., Karatzas and Shreve, 1998, p. 103) completes the proof.

A.3 Derivation of (25) and (29)

This appendix writes the individual's consumption choice c_t^* in terms of unexpected past stock return shocks. We can write the stochastic discount factor $\widehat{M}_t = M_t x^{Q_t - 1}(1+\beta P_t)$ as follows (this follows from applying Itô's lemma to $\widehat{M}_t = f(M_t, P_t, Q_t) = M_t x^{Q_t-1}(1+\beta P_t)$):

$$\widehat{M}_t = \widehat{M}_0 \exp\left\{ -\int_0^t \left(\widehat{r}_s + \frac{1}{2}\lambda^2 \right) \mathrm{d}s \right\} \exp\left\{ -\lambda \int_0^t \mathrm{d}W_s \right\}, \tag{60}$$

where

$$\widehat{r}_s = \beta + \frac{\widetilde{r}_s - \alpha\beta P_s}{1+\beta P_s} \tag{61}$$

with $\widetilde{r}_s = r - \beta e^{-(\alpha-\beta)s} \log x$.

Substituting (60) into (24), we arrive at

$$\widehat{c}_t^* = \frac{c_t^*}{h_t^*} = \exp\left\{ \frac{1}{\gamma} \int_0^t \left(\widehat{r}_s + \frac{1}{2}\lambda^2 - \delta \right) \mathrm{d}s + \frac{\bar{y}}{\gamma} \right\} \exp\left\{ \frac{\lambda}{\gamma} \int_0^t \mathrm{d}W_s \right\}. \tag{62}$$

Here, $\bar{y} = -\left(\log y + \log \widehat{M}_0 \right)$.

We can write the habit level h_t^* as follows:

$$\begin{aligned}
h_t^* &= \exp\left\{ \int_0^t \beta \exp\{-(\alpha-\beta)(t-s)\} \log \widehat{c}_s^{\,*} \mathrm{d}s \right\} \\
&= \exp\left\{ \int_0^t \beta \exp\{-(\alpha-\beta)(t-s)\} \right. \\
&\qquad\qquad \left. \left[\frac{1}{\gamma} \int_0^s \left(\widehat{r}_u + \frac{1}{2}\lambda^2 - \delta \right) \mathrm{d}u + \frac{\bar{y}}{\gamma} + \frac{\lambda}{\gamma} \int_0^s \mathrm{d}W_u \right] \mathrm{d}s \right\} \\
&= \exp\left\{ \int_0^t \left(\frac{1}{\gamma} Q_{t-s} - \frac{1}{\gamma} \right) \left(\widehat{r}_s + \frac{1}{2}\lambda^2 - \delta \right) \mathrm{d}s \right\} \\
&\quad \times \exp\left\{ \left(\frac{1}{\gamma} Q_t - \frac{1}{\gamma} \right) \bar{y} + \int_0^t \left(\frac{\lambda}{\gamma} Q_{t-s} - \frac{\lambda}{\gamma} \right) \mathrm{d}W_s \right\}.
\end{aligned} \tag{63}$$

Hence,

$$
\begin{aligned}
c_t^* &= h_t^* \exp\left\{\frac{1}{\gamma}\int_0^t \left(\widehat{r}_s + \frac{1}{2}\lambda^2 - \delta\right) \mathrm{d}s + \frac{\bar{y}}{\gamma}\right\} \exp\left\{\frac{\lambda}{\gamma}\int_0^t \mathrm{d}W_s\right\} \\
&= \exp\left\{\frac{1}{\gamma}\int_0^t Q_{t-s}\left(\widehat{r}_s + \frac{1}{2}\lambda^2 - \delta\right) \mathrm{d}s + \frac{1}{\gamma}Q_t\bar{y} + \frac{\lambda}{\gamma}\int_0^t Q_{t-s}\,\mathrm{d}W_s\right\} \\
&= (c_0^*)^{Q_t} \exp\left\{\frac{1}{\gamma}\int_0^t Q_{t-s}\left(\widehat{r}_s + \frac{1}{2}\lambda^2 - \delta\right) \mathrm{d}s + \frac{\lambda}{\gamma}\int_0^t Q_{t-s}\,\mathrm{d}W_s\right\}.
\end{aligned}
\tag{64}
$$

It follows from (64) that

$$
q_{t-s} = \frac{\lambda}{\gamma\sigma} Q_{t-s} \tag{65}
$$

models the sensitivity of log consumption $\log c_t^*$ to the unexpected stock return shock $\sigma\,\mathrm{d}W_s$.

Subtracting $\log c_{t+h}^*$ from $\log c_t^*$ and taking the limit $h \to 0$, we arrive at (29).

A.4 Proof of Theorem 4.2

Straightforward computations show that

$$
\begin{aligned}
V_{t,h} &= \mathbb{E}_t\left[\frac{M_{t+h}}{M_t} c_{t+h}^*\right] \\
&= c_t^* G_{t,h} \mathbb{E}_t\left[\exp\left\{-\int_0^h \left(r + \frac{1}{2}\lambda^2\right) \mathrm{d}v - \lambda\int_0^h \mathrm{d}W_{t+h-v}\right\}\right. \\
&\qquad \left.\times\ \exp\left\{\frac{1}{\gamma}\int_0^h Q_v\left(\widehat{r}_{t+h-v} + \frac{1}{2}\lambda^2 - \delta\right) \mathrm{d}v + \frac{\lambda}{\gamma}\int_0^h Q_v\,\mathrm{d}W_{t+h-v}\right\}\right] \\
&= c_t^* G_{t,h} C_{t,h},
\end{aligned}
\tag{66}
$$

where

$$
C_{t,h} = \exp\left\{-\int_0^h \left(r - Q_v\frac{1}{\gamma}\left[\widehat{r}_{t+h-v} + \frac{1}{2}\lambda^2 - \delta\right] + Q_v\frac{\lambda^2}{\gamma} - \frac{1}{2}Q_v^2\frac{\lambda^2}{\gamma^2}\right) \mathrm{d}v\right\}, \tag{67}
$$

$$
\begin{aligned}
G_{t,h} &= (c_0^*)^{(Q_{t+h}-Q_t)} \exp\left\{\frac{1}{\gamma}\int_0^t (Q_{t+h-s} - Q_{t-s})\left(\widehat{r}_s + \frac{1}{2}\lambda^2 - \delta\right) \mathrm{d}s\right\} \\
&\qquad \times \exp\left\{\frac{\lambda}{\gamma}\int_0^t (Q_{t+h-s} - Q_{t-s})\,\mathrm{d}W_s\right\}.
\end{aligned}
\tag{68}
$$

Eqn. (66) shows that the term $V_{t,h}/c_t^*$ consists of two factors. The factor $G_{t,h}$ represents past stock return shocks that the individual absorbs into future growth rates of (median) consumption. This factor equals unity if the individual directly absorbs unexpected stock returns shocks into current consumption. The factor $C_{t,h}$ summarizes the impacts of the unconditional growth rates of median consumption and the future (uncertain) rates of return on the market value of future consumption.

It follows from Itô's lemma that $\log V_t = \log\left[\int_0^{T-t} V_{t,h}\,\mathrm{d}h\right]$ satisfies

$$\mathrm{d}\log V_t = (\ldots)\,\mathrm{d}t + \frac{\lambda}{\gamma}\int_0^{T-t} Q_h \frac{V_{t,h}}{V_t}\,\mathrm{d}h \cdot \mathrm{d}W_t, \tag{69}$$

suppressing the drift term for brevity. It also holds that (this follows from applying Itô's lemma to the dynamic budget constraint (7))

$$\mathrm{d}\log A_t = (\ldots)\,\mathrm{d}t + \sigma \cdot \frac{\pi_t}{A_t} \cdot \mathrm{d}W_t. \tag{70}$$

Setting Eqn. (70) equal to Eqn. (69) and solving for the approximate optimal portfolio choice, we arrive at (32).

A.5 Proof of Theorem 5.1

We first write the individual's consumption choice c_t^* in terms of unexpected past stock return and interest rate shocks. The stochastic discount factor $\widehat{M}_t = M_t\,(1+\beta P_t)$ is given by (this follows from applying Itô's lemma to $\widehat{M}_t = f\,(M_t, P_t) = M_t\,(1+\beta P_t)$):[43]

$$\widehat{M}_t = \widehat{M}_0 \exp\left\{-\int_0^t \left(\widehat{r}_s + \frac{1}{2}\left|\left|\widehat{\lambda}_s\right|\right|^2\right)\mathrm{d}s\right\} \exp\left\{-\widehat{\lambda}_s^\top \int_0^t \mathrm{d}W_s\right\}, \tag{71}$$

where

$$\widehat{r}_s = \beta + \frac{r_s - \alpha\beta P_s}{1+\beta P_s}, \tag{72}$$

$$\widehat{\lambda}_{1,s} = \lambda_1 + \beta\frac{\sigma_r \rho \widehat{D}_s P_s}{1+\beta P_s}, \tag{73}$$

$$\widehat{\lambda}_{2,s} = \lambda_2 + \beta\frac{\sigma_r\sqrt{1-\rho^2}\,\widehat{D}_s P_s}{1+\beta P_s}, \tag{74}$$

with

$$\widehat{D}_s = \int_0^{T-s} \alpha_{s,h} D_h\,\mathrm{d}h. \tag{75}$$

[43]For the sake of simplicity, we assume $x = 1$.

Here,

$$D_h = \frac{1 - \exp\{-\kappa h\}}{\kappa}, \tag{76}$$

$$\alpha_{s,h} = \frac{e^{-\int_0^h \left(\alpha - \beta + r_s + \kappa D_u (\bar{r} - r_s) - \sigma_r D_u \left(\lambda_1 \rho + \lambda_2 \sqrt{1-\rho^2}\right) - \frac{1}{2}\sigma_r^2 D_u^2\right) \mathrm{d}u}}{\int_0^{T-s} e^{-\int_0^h \left(\alpha - \beta + r_s + \kappa D_u (\bar{r} - r_s) - \sigma_r D_u \left(\lambda_1 \rho + \lambda_2 \sqrt{1-\rho^2}\right) - \frac{1}{2}\sigma_r^2 D_u^2\right) \mathrm{d}u} \mathrm{d}h}. \tag{77}$$

Substituting (71) into (24), we arrive at

$$\widehat{c}_t^{\,*} = \frac{c_t^*}{h_t^*} = \exp\left\{\frac{1}{\gamma}\int_0^t \left(\widehat{r}_s + \frac{1}{2}\left|\left|\widehat{\lambda}_s\right|\right|^2 - \delta\right) \mathrm{d}s + \frac{\bar{y}}{\gamma}\right\} \exp\left\{\frac{1}{\gamma}\widehat{\lambda}_s^\top \int_0^t \mathrm{d}W_s\right\}. \tag{78}$$

Here, $\bar{y} = -\left(\log y + \log \widehat{M}_0\right)$.

We express the habit level h_t^* as follows:

$$\begin{aligned} h_t^* &= \exp\left\{\int_0^t \beta \exp\{-(\alpha-\beta)(t-s)\} \log \widehat{c}_s^{\,*} \mathrm{d}s\right\} \\ &= \exp\left\{\int_0^t \left(\frac{1}{\gamma}Q_{t-s} - \frac{1}{\gamma}\right)\left(\widehat{r}_s + \frac{1}{2}\left|\left|\widehat{\lambda}_s\right|\right|^2 - \delta\right) \mathrm{d}s\right\} \\ &\quad \times \exp\left\{\left(\frac{1}{\gamma}Q_t - \frac{1}{\gamma}\right)\bar{y} + \int_0^t \left(\frac{1}{\gamma}Q_{t-s}\widehat{\lambda}_s^\top - \frac{1}{\gamma}\widehat{\lambda}_s^\top\right) \mathrm{d}W_s\right\}. \end{aligned} \tag{79}$$

Hence,

$$\begin{aligned} c_t^* &= h_t^* \exp\left\{\frac{1}{\gamma}\int_0^t \left(\widehat{r}_s + \frac{1}{2}\left|\left|\widehat{\lambda}_s\right|\right|^2 - \delta\right) \mathrm{d}s + \frac{\bar{y}}{\gamma}\right\} \exp\left\{\frac{1}{\gamma}\widehat{\lambda}_s^\top \int_0^t \mathrm{d}W_s\right\} \\ &= (c_0^*)^{Q_t} \exp\left\{\frac{1}{\gamma}\int_0^t Q_{t-s}\left(\widehat{r}_s + \frac{1}{2}\left|\left|\widehat{\lambda}_s\right|\right|^2 - \delta\right) \mathrm{d}s + \frac{1}{\gamma}\int_0^t Q_{t-s}\widehat{\lambda}_s^\top \mathrm{d}W_s\right\}. \end{aligned} \tag{80}$$

The market value at time t of the future consumption stream $\{c_s^*\}_{t \le s \le T}$, i.e., $V_t = \int_0^{T-t} V_{t,h}\,\mathrm{d}h$, is a function of the state variables r_t and $\log \widehat{M}_t$. It now follows from Itô's lemma that

$$\begin{aligned} \mathrm{d}\log V_t = (\ldots)\,\mathrm{d}t &- \left(\widehat{\lambda}_{1,t}\frac{\partial V_t}{\partial \log \widehat{M}_t}\frac{1}{V_t} - \sigma_r \rho \frac{\partial V_t}{\partial r_t}\frac{1}{V_t}\right)\mathrm{d}W_{1,t} \\ &- \left(\widehat{\lambda}_{2,t}\frac{\partial V_t}{\partial \log \widehat{M}_t}\frac{1}{V_t} - \sigma_r \sqrt{1-\rho^2}\frac{\partial V_t}{\partial r_t}\frac{1}{V_t}\right)\mathrm{d}W_{2,t}. \end{aligned} \tag{81}$$

It also holds that

$$\mathrm{d}\log A_t = (\dots)\,\mathrm{d}t + \left(\frac{\pi_{1,t}}{A_t}\sigma_S - \frac{\pi_{2,t}}{A_t}\sigma_r \rho D_{T_1}\right)\mathrm{d}W_{1,t} - \frac{\pi_{2,t}}{A_t}\sigma_r\sqrt{1-\rho^2}D_{T_1}\,\mathrm{d}W_{2,t}. \tag{82}$$

Setting Eqn. (82) equal to Eqn. (81) and solving for the approximate optimal portfolio choice, we arrive at (37) and (38).

A.6 Proof of Theorem 6.1

Given $\widehat{A}_0$, the approximate optimal relative consumption choice $\widehat{c}_t^*$ can be obtained from Schroder and Skiadas (1999). Finally, the approximate optimal consumption choice c_t^* follows as in Eqn. (17).

B Uncertain Date of Death

So far we have assumed that the terminal time T is known at the beginning of the life cycle. However, the individual may also want to know how to drawdown his accumulated wealth if the terminal time T is equal to his uncertain date of death. This appendix explores how an uncertain terminal time affects the consumption dynamics (29). We assume that the individual aims to maximize lifetime utility (23) where $T \geq 0$ now denotes the *uncertain* adult age at which the individual passes away.

We find that in this setting of uncertain terminal time, the individual's log consumption choice $\log c_t^*$ evolves according to

$$\mathrm{d}\log c_t^* = \widetilde{g}_t\,\mathrm{d}t + \widetilde{p}_t\,\mathrm{d}t + \frac{\lambda}{\gamma\sigma}\sigma\,\mathrm{d}W_t, \tag{83}$$

which is to be compared to (29). Here,

$$\widetilde{g}_t = \frac{1}{\gamma}\left(\widehat{r}_t + \frac{1}{2}\lambda^2 - \delta - H_t\right), \tag{84}$$

$$\widetilde{p}_t = Q_t' \log c_0^* + \frac{1}{\gamma}\int_0^t Q_{t-s}'\widetilde{g}_s\,\mathrm{d}s + \frac{\lambda}{\gamma\sigma}\int_0^t Q_{t-s}'\sigma\,\mathrm{d}W_s, \tag{85}$$

with $\widehat{r}_t = \beta + (r - \alpha\beta P_t)\,/\,(1+\beta P_t)$, H_t the force of mortality (hazard rate) at adult age t, $Q_{t-s}' = \mathrm{d}Q_{t-s}/\,\mathrm{d}t$, and P_t and Q_{t-s} defined in (20) and (26), respectively.[44]

[44]We assume no uncertainty in the force of mortality.

As shown by Eqn. (83), an increase in future consumption now implies two types of costs. First, the individual prefers to consume sooner rather than later. This effect is captured by the time preference rate δ. Second, the individual may pass away before being able to enjoy future consumption. This effect is captured by the force of mortality H_t. As a result, the median consumption path is less steep compared to the case where the terminal time T is assumed to be fixed; see Figure 10. In this figure, we compute the force of mortality using the unisex mortality table for the US population for 2015.

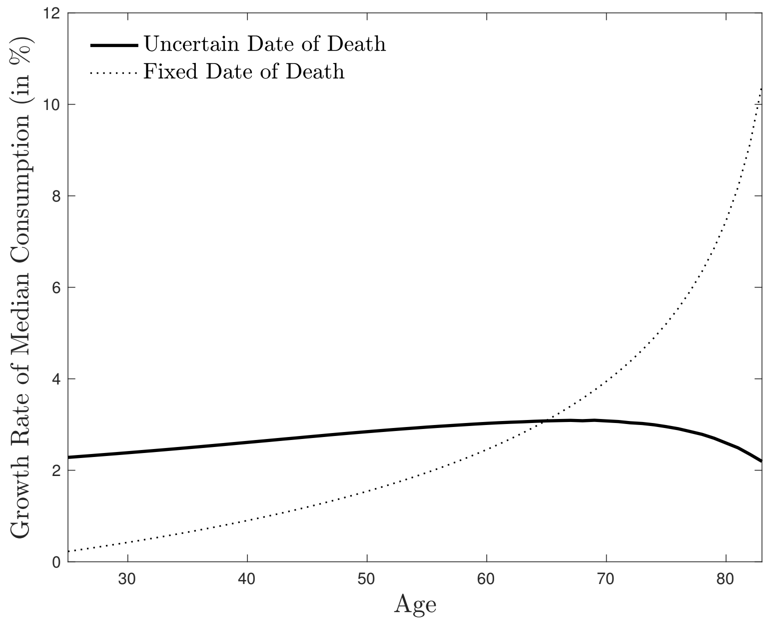


Figure 10: **Growth rate of median consumption.** The figure illustrates the growth rate of median consumption as a function of age for the case where the terminal time T is equal to the individual's uncertain date of death. The individual preferences exhibit internal habit formation (with preference parameters $\gamma = 10$, $\alpha = 0.3$, and $\beta = 0.3$). Survival rates are taken from the Human Mortality Database. We use the unisex mortality table for the US population for 2015. Wealth at the age of 25 is equal to 45. For comparison purposes, we also plot the growth rate of median consumption for the case with a fixed date of death; see the dotted line. We set the risk-free interest rate r equal to 1%, the market price of risk λ to 0.2, the stock return volatility σ to 20%, and the subjective rate of time preference δ to 3%. The individual adjusts consumption once a year.

B.1 Proof of (83)

The individual's approximate optimization problem is given by

$$\begin{aligned} \max_{\widehat{c}_t : 0 \le t \le T_{\max}} \quad & \mathbb{E}\left[\int_0^{T_{\max}} e^{-\delta t} e^{-\int_0^t H_s \, \mathrm{d}s} \frac{1}{1-\gamma} \left(\widehat{c}_t\right)^{1-\gamma} \mathrm{d}t\right] \\ \text{s.t.} \quad & \mathbb{E}\left[\int_0^{T_{\max}} \widehat{M}_t \widehat{c}_t \, \mathrm{d}t\right] \le \widehat{A}_0. \end{aligned} \tag{86}$$

Here, $T_{\max}$ denotes the maximum adult age the individual can reach.

The Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} &= \mathbb{E}\left[\int_0^{T_{\max}} e^{-\delta t} e^{-\int_0^t H_s \, \mathrm{d}s} \frac{1}{1-\gamma} \left(\widehat{c}_t\right)^{1-\gamma} \mathrm{d}t\right] + y\left(\widehat{A}_0 - \mathbb{E}\left[\int_0^{T_{\max}} \widehat{M}_t \widehat{c}_t \, \mathrm{d}t\right]\right) \\ &= \int_0^{T_{\max}} \mathbb{E}\left[e^{-\delta t} e^{-\int_0^t H_s \, \mathrm{d}s} \frac{1}{1-\gamma} \left(\widehat{c}_t\right)^{1-\gamma} - y \widehat{M}_t \widehat{c}_t\right] \mathrm{d}t + y \widehat{A}_0. \end{aligned} \tag{87}$$

Here, $y \ge 0$ denotes the Lagrange multiplier associated with the static budget constraint. The individual aims to maximize $e^{-\delta t} e^{-\int_0^t H_s \, \mathrm{d}s} \frac{1}{1-\gamma} \left(\widehat{c}_t\right)^{1-\gamma} - y \widehat{M}_t \widehat{c}_t$. The approximate optimal relative consumption choice $\widehat{c}_t^*$ satisfies the following first-order optimality condition:

$$e^{-\delta t} e^{-\int_0^t H_s \, \mathrm{d}s} \left(\widehat{c}_t^*\right)^{-\gamma} = y \widehat{M}_t. \tag{88}$$

After solving the first-order optimality condition, we obtain the following maximum:

$$\widehat{c}_t^* = \left(e^{\delta t} e^{\int_0^t H_s \, \mathrm{d}s} y \widehat{M}_t\right)^{-\frac{1}{\gamma}}. \tag{89}$$

Hence (use (17)),

$$c_t^* = h_t^* \left(y e^{\delta t} e^{\int_0^t H_s \, \mathrm{d}s} \widehat{M}_t\right)^{-\frac{1}{\gamma}}. \tag{90}$$

We can now derive the consumption dynamics (83) similarly as in the proof of (29).

C Excessive Median Growth Rates of Consumption

We state the following theorem.

Theorem C.1. *Suppose that r_t is constant (i.e., $r_t = r$) and let $\widehat{r}_t$ be defined as follows:*

$$\widehat{r}_t = \beta + \left(r - \alpha\beta P_t\right) / \left(1 + \beta P_t\right). \tag{91}$$

Then:

1. *The value of* $\widehat{r}_t$ *increases as the preference parameter* β *increases, given fixed* $\alpha - \beta$.
2. *The value of* $\widehat{r}_t$ *decreases as the terminal time* T *increases. In particular,* $\widehat{r}_t \to r$ *if* $T \to \infty$.

Theorem C.1 and the decomposition in (29) imply that current consumption has a large impact on future habit levels if the preference parameter β is large. Also, the utility gain of an increase in consumption is smaller when the individual is (relatively) young (i.e., small t) than when the individual is (relatively) old (i.e., large t). As a result, an individual with habit preferences prefers (unrealistically) high unconditional median growth rates of log consumption (especially at high ages) except when his subjective rate of time preference δ is excessive.

C.1 Proof of Theorem C.1

We first prove that the (partial) derivative of $\widehat{r}_t$ with respect to β is positive given fixed $\alpha - \beta$.[45] Define $\eta = \alpha - \beta$. Substituting $\alpha = \eta + \beta$ into (91), we find

$$\widehat{r}_t = \beta + \frac{r - (\eta + \beta)\beta P_t}{1 + \beta P_t}. \tag{92}$$

The (partial) derivative of $\widehat{r}_t$ with respect to β is given by

$$\begin{aligned} \frac{\partial \widehat{r}_t}{\partial \beta} &= 1 + \frac{-\left(1 + \beta P_t\right)\left(\eta + 2\beta\right) P_t - \left(r - (\eta + \beta)\beta P_t\right) P_t}{\left(1 + \beta P_t\right)^2} \\ &= 1 + \frac{-\eta P_t - 2\beta P_t - \eta \beta P_t^2 - 2\left(\beta P_t\right)^2 - r P_t + \eta \beta P_t^2 + \left(\beta P_t\right)^2}{1 + 2\beta P_t + \left(\beta P_t\right)^2} \\ &= 1 + \frac{-\eta P_t - 2\beta P_t - \left(\beta P_t\right)^2 - r P_t}{1 + 2\beta P_t + \left(\beta P_t\right)^2}. \end{aligned} \tag{93}$$

Hence,

$$\begin{aligned} \frac{\partial \widehat{r}_t}{\partial \beta} \geq 0 &\Leftrightarrow \frac{-\eta P_t - 2\beta P_t - \left(\beta P_t\right)^2 - r P_t}{1 + 2\beta P_t + \left(\beta P_t\right)^2} \geq -1 \\ &\Leftrightarrow \eta P_t + 2\beta P_t + \left(\beta P_t\right)^2 + r P_t \leq 1 + 2\beta P_t + \left(\beta P_t\right)^2 \\ &\Leftrightarrow \left(r + \eta\right) P_t \leq 1 \\ &\Leftrightarrow 1 - \exp\left\{-(r + \eta)(T - t)\right\} \leq 1. \end{aligned} \tag{94}$$

[45] In the derivation of Theorem C.1, we assume $x = 1$, so that $\widetilde{r}_t = r$ for all t.

Hence, $\partial \widehat{r}_t / \partial \beta$ is positive given fixed $\alpha - \beta$.

Finally, we prove that the (partial) derivative of $\widehat{r}_t$ with respect to T is negative. The (partial) derivative of $\widehat{r}_t$ with respect to T is given by

$$\frac{\partial \widehat{r}_t}{\partial T} = -r \left(1 + \beta P_t\right)^{-2} \frac{\partial P_t}{\partial T} - \alpha\beta \left(1 + \beta P_t\right)^{-2} \frac{\partial P_t}{\partial T}. \tag{95}$$

Using the fact that $\partial P_t / \partial T$ is positive, we find that $\partial \widehat{r}_t / \partial T$ is negative. Furthermore, simple algebra yields that $\widehat{r}_t = r$ if $T = \infty$. Here, we use the fact that $P_t \to 1/(r + \alpha - \beta)$ as $T \to \infty$.

D Numerical Solution Method

To assess the accuracy of our pathwise approximation, we also determine the genuine optimal consumption and portfolio policies using numerical backward induction. Because we only explore the case $\alpha = \beta$, we can reduce the number of state variables from two (i.e., wealth level and habit level) to one (i.e., wealth-to-habit ratio). The first step is to specify discrete points in the state space, called grid points. For each grid point, we determine the optimal relative consumption choice and the optimal portfolio choice. To determine the optimal policies, we need to evaluate the utility value for every combination of relative consumption choice and portfolio choice. The utility value is equal to the sum of current utility and the discounted expected continuation value. Once we have computed the utility value for every combination of relative consumption choice and portfolio choice, we select the maximum utility value. We then use this maximum utility value to solve the previous period's maximization problem. This process is iterated backwards in time until the entire life-cycle problem has been solved. In the last period, the optimal relative consumption choice and the maximum utility value are given by $\widehat{c}_T^{\text{opt}} = A_T / h_T$ and $\left(\widehat{c}_T^{\text{opt}}\right)^{1-\gamma} / (1 - \gamma)$, respectively. This gives us the terminal condition for the backward induction procedure. We use Gaussian quadrature to compute expectations. For points that do not lie on the state space grid, we evaluate the utility level using cubic spline interpolation.

We now introduce the following notation:

- $\mathcal{S}$: total number of simulations;
- Δt: time step;
- $t_n = n\Delta t$ for $n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor$.

The floor operator $\lfloor \cdot \rfloor$ rounds a number downward to its nearest integer.

To compute the welfare loss associated with the approximate consumption strategy, we apply the following steps:

1. We generate $\mathcal{S}$ trajectories of the stochastic discount factor ($s = 1, \ldots, \mathcal{S}$):

$$M_{s,t_{n+1}} = M_{s,t_n} - rM_{s,t_n}\Delta t - \lambda M_{s,t_n}\sqrt{\Delta t}\epsilon_{s,t_n}, \quad n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor. \tag{96}$$

   Here, ϵ_{s,t_n} is a standard normally distributed random variable.
2. We compute the approximate relative consumption choice $\widehat{c}^*_{s,t_n}$ and the approximate portfolio strategy π^*_{s,t_n} for $s = 1, \ldots, \mathcal{S}$ and $n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor$. We note that the approximate relative consumption choice $\widehat{c}^*_{s,t_n}$ is a function of the stochastic discount factor $\widehat{M}_{s,t_n} = M_{s,t_n}\left(1 + \beta P_{t_n}\right)$. The individual's lifetime utility $U\left(c/h\right)$ can now be obtained by using the method of numerical backward induction. Note that in this step we use backward induction only to obtain lifetime utility (we do not use it to obtain the optimal solutions).
3. We numerically solve for the certainty equivalent consumption ce^*.
4. We compute the optimal consumption strategy c^{opt}_{s,t_n} and the optimal portfolio strategy $\pi^{\mathrm{opt}}_{s,t_n}$ for $s = 1, \ldots, \mathcal{S}$ and $n = 0, \ldots, \left\lfloor \frac{T}{\Delta t} \right\rfloor$. Lifetime utility follows from the backward induction algorithm.
5. We numerically solve for the optimal certainty equivalent consumption ce^{opt}.
6. Finally, we compute the welfare loss l:

$$l = \frac{ce^{\mathrm{opt}} - ce^*}{ce^{\mathrm{opt}}}. \tag{97}$$

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