The Duration Puzzle in Life-Cycle Investment[*]

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Abstract

By analyzing the portfolio allocations of Target Date Funds (TDFs), we document that the observed durations of TDF portfolios are *in*consistent with the durations predicted by classical portfolio theory. We call this stylized fact the duration puzzle. We investigate to what extent several extensions of classical portfolio theory can explain the duration puzzle. More specifically, we consider the impact of human capital, inflation risk and portfolio restrictions on the duration of the optimal portfolio. We find that it is difficult to explain the duration puzzle, especially for individuals aged between 35 and 65.

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1 Introduction

It is well-known since Merton (1973) that time-variation in investment opportunities has important implications for the optimal asset allocation of long-term investors.[1] Instead of just caring about how much an asset contributes to the portfolio's short-term return and variance, Merton (1973) shows that long-term investors should worry about an asset's correlation with state variables that describe next period's investment opportunities. One particularly clear example of such a state variable is the interest rate. For long-term investors, a decline in the interest rate is bad news since future returns on many asset classes are positively correlated with interest rate movements. Indeed, the risk-free asset for a long-term investor is not cash, but an (inflation-linked) bond with a maturity that corresponds to the investor's investment horizon (Wachter (2003)).

An investor saving for retirement should thus continuously rebalance her optimal portfolio mix. However, empirical research shows that investors suffer from inertia: investors make few active changes to the portfolio mixes selected for them by the employer; see, e.g., Madrian and Shea (2001). Therefore, so-called Target Date Funds (TDFs) are often used as default funds. These funds offer a managed portfolio strategy that should remain appropriate to an investor's investment horizon even if left unreviewed. In the absence of market imperfections and perverse incentives, we would expect the TDF manager to provide a portfolio strategy that is suitable for the 'average' TDF investor saving for retirement. From a theoretical perspective, one would expect then that the interest rate duration of a TDF portfolio decreases with the investor's age (see, e.g., Campbell and Viceira (2001) and Brennan and Xia (2002)). In this paper, we document that this is not the case. We call the stylized fact that observed durations of TDF portfolios are *not* consistent with the theoretically predicted durations the *duration puzzle*. We show that several extensions of classical portfolio theory fail to explain the duration puzzle.

We obtain panel data on TDFs from Morningstar. By combining the portfolio allocations of TDFs with different target dates, we construct the portfolio allocation over the investor's life-cycle. Figure 1 shows the average duration of the fixed income portfolios held by TDFs in 2019 as a function of age, assuming that the target date corresponds to a retirement age of 65. The figure also shows the 10% and 90% quantiles.

[1] Since Merton (1973), many authors have studied the implications of stochastic investment opportunities for the optimal asset allocation over the investor's life-cycle; see, e.g., Campbell, Cocco, Gomes, Maenhout, and Viceira (2001), Wachter (2002), Chacko and Viceira (2005), Liu (2007), Munk (2008), and Buraschi, Porchia, and Trojani (2010).

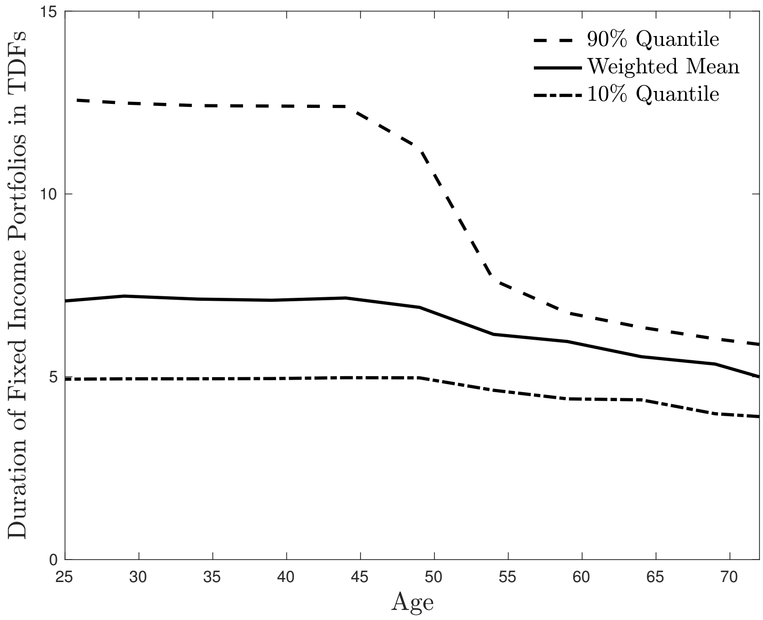


Figure 1. Duration of fixed income portfolios in TDFs. The figure illustrates the (weighted) mean (solid line), the (weighted) 10% quantile (dash-dotted line) and the (weighted) 90% quantile (dashed line) of the modified duration of fixed income portfolios in TDFs in 2019 as a function of age. The age on the horizontal axis is chosen such that the target date corresponds to age 65. Note that a TDF typically remains open when the target date is reached.

Figure 1 reveals a striking pattern: the average duration of the fixed income portfolios in TDFs is flat over the life-cycle and, furthermore, is low relative to the investor's investment horizon. This pattern seems to be at odds with classical portfolio theory. As pointed out by e.g., Campbell and Viceira (2001), bonds with a low duration are an undesirable asset class from the perspective of a long-term investor. Indeed, short-term bonds on average offer a lower yield compared to long-term bonds, and, moreover, expose *long-term* investors to interest rate risk. Campbell and Viceira (2001) conclude that aggressive long-term investors should hold more stocks, while conservative long-term investors should hold more long-term bonds, not short-term bonds. However, we find that TDF managers invest in bonds with a relatively low duration. Additionally, we do not observe any (strong) interaction with the investor's investment horizon.

The extent to which investors are hedged against interest rate risk depends on the duration of the *total* TDF assets. Hence, this duration depends not only on the duration of the fixed income portfolio but also on the interest rate duration of the stock portfolio. Unfortunately, the data does not allow us to adequately measure the interest rate duration of the stock portfolios. If we assume a stock portfolio duration of 5, which is consistent with, e.g., Bernanke and Kuttner (2005), we find that the average duration of the total TDF assets is flat over the life-cycle and never exceeds five.

The patterns found in the data raise several questions. Since TDFs were invented on the idea that investors' investment horizons matter, we ask ourselves the question what could explain these patterns. Why does a TDF manager choose the same duration of the total TDF assets for young and old individuals? Why is the duration of the total TDF assets so low in general, even for older individuals? We explore to what extent a rational life-cycle investment model can provide answers to these questions. Our point of departure is the classical investment model of Brennan and Xia (2002). These authors solve a consumption and portfolio choice problem in a setting with equity risk, real interest rate risk and inflation risk. They find that the duration of the optimal investment portfolio is a monotonically decreasing function of the investment horizon, which is at odds with the data.

From this starting point, we consider several alternative model specifications to explain the duration puzzle. We start by extending the model of Brennan and Xia (2002) to allow for labor income. Particularly early in life, the present value of labor income (i.e., human capital) makes up a significant part of total wealth. Since human capital is a long-lived asset, one would expect that its presence in the investor's total wealth portfolio will interact with the optimal shares of financial wealth invested in stocks and long-term bonds.[2] Adding human capital to the model does indeed change the optimal bond holdings. However, it can not explain the observed patterns seen in the data. We find that the duration of the optimal financial wealth portfolio for an investor with finite risk aversion is no longer monotonically decreasing with age but becomes somewhat hump shaped over the life-cycle. Intuitively, early in life, the degree of interest rate risk hedging provided by the investor's human capital – which can be viewed as a long-term bond – is sufficient. Later in life however, as the duration of human capital declines faster than the duration of remaining lifetime consumption, the investor starts to increase the duration of her financial wealth portfolio. After a certain age, when the investor's remaining expected lifetime is short and interest rate risk hedging is not very important anymore, the investor begins to reduce the duration of her financial wealth portfolio. Also, we find that individuals with a relatively high

[2]See, e.g., Bodie, Merton, and Samuelson (1992) and Benzoni, Collin-Dufresne, and Goldstein (2007) for long-term portfolio choice models with labor income. These papers, however, do not feature interest rate risk.

retirement income from an outside source (e.g., social security) have a relatively low demand for long-term bonds. Indeed, if total future income consists for a large part of social security, the individual is already (partially) hedged against interest rate declines.

Then, we study the impact of inflation risk on the duration of the optimal investment portfolio. First, we consider the case where nominal as well as inflation-linked bonds are available in the financial market. The investor invests in the nominal bond for a speculative reason: she wants to profit from the inflation risk premium. She also buys inflation-linked bonds to hedge any remaining real interest rate risk. The duration of her overall investment portfolio does not change as result of adding inflation risk to the model. We also consider the case where inflation-linked bonds are not available. If the financial market for inflation risk hedging is incomplete, investors face a trade off between hedging their nominal interest rate risk versus hedging their inflation risk. Nominal long-term bonds provide a hedge against fluctuations in the nominal interest rate but expose investors to inflation risk. Campbell and Viceira (2001) show that an investor may significantly reduce the duration of the optimal investment portfolio if inflation shocks are highly persistent. We find that, while the presence of unhedgeable inflation risk indeed leads to a lower duration of the bond portfolio, it remains difficult to explain the duration puzzle, unless we assume that uncertainty about future inflation rates is significantly larger than real interest rate uncertainty.

Finally, we consider the role of portfolio restrictions. Portfolio allocations predicted by rational life-cycle investment models may be extremely leveraged (i.e., one should borrow money to invest in the financial market). In practice, however, it is unlikely that TDF managers take highly leveraged positions. The consequence of imposing portfolio restrictions is that risky stocks and long-term bonds compete for the scarce available resources to invest in the financial market. This may explain the limited role of long-term bonds in TDF portfolios, especially for young investors whose financial wealth is only a small portion of total wealth. This still cannot explain why the fraction of financial wealth invested in fixed income assets is primarily invested in short-term bonds. The scarcity of available resources would actually be a stimulus for young investors to invest in bonds with a high maturity.

Overall, we conclude that it is hard to reconcile observed durations of TDF portfolios with a rational life-cycle investment model.

2 Empirical Life-Cycle Paths

This section explores the life-cycle paths implemented by Target Date Funds (TDFs). TDFs are mutual funds that are specifically designed to provide individuals a portfolio allocation that is dynamically optimized for a specific investment horizon. TDFs have become increasingly popular since the introduction of the Pension Protection Act 2006. This regulation created an incentive for employers to make TDFs the default investment option in 401(k) pension plans, by labeling them as a Qualified Default Investment Alternative (QDIA). According to the act, employers will no longer be liable from losses that result from investments in a QDIA. In 2016, 79.5% of large 401(k) pension plans offered TDFs, 75% of 401(k) plan participants were offered TDFs, and more than 50% of 401(k) plan participants held assets in TDFs (Investment Company Institute (2019)). Furthermore, the share of 401(k) assets invested in TDFs grew from 5% in 2006 to 21% in 2016.[3]

We obtain panel data on TDFs from Morningstar for the years 2017 and 2019. For 2017, the data contains 599 TDFs and for 2019 this number equals 524. TDFs typically come in series with 5 year target date intervals. For example, the *BlackRock LifePath Index Series* is a series of TDFs with target dates 2020, 2025, 2030, 2035, 2040, 2045, 2050, 2055 and 2060. Often a TDF series also includes a fund labeled as 'retirement fund'. For 2017, the data covers 64 TDF series and for 2019 we observe 61 TDF series.

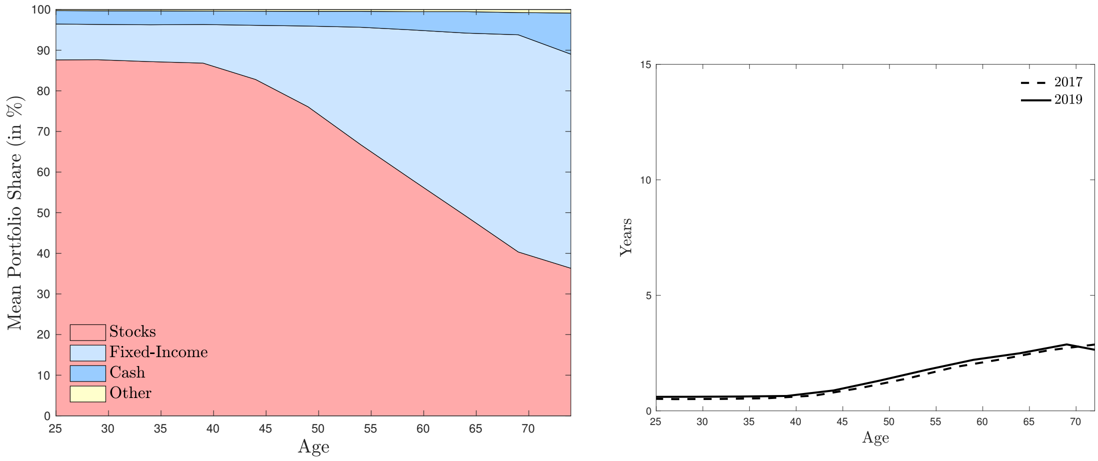
In 2017, total assets under management by TDFs in our dataset amount to 900 billion USD. This means that the dataset roughly covers the total value of the U.S. TDF market (Investment Company Institute (2019)). In 2019, total assets under management in our dataset equal 1200 billion USD. The two largest TDF providers are Vanguard and Fidelity with 450 and 250 billion USD of assets under management in 2019, respectively.

For each TDF series we construct the implied life-cycle investment strategy. We assume that the target date corresponds to a retirement age of 65. We then determine how the asset allocation changes over the life-cycle. Figure 2(a) shows the (weighted) mean shares of TDF assets invested in stocks, fixed income instruments, cash and other instruments as a function of age based on 2019 data. The figure shows that the asset allocation policy reduces the stock exposure as investors age. This is in line with standard life-cycle theory; see Bodie et al. (1992) and Cocco, Gomes, and Maenhout (2005).

[3]Another important asset category in an investor's retirement portfolio that provides a hedge against interest rate risk is a directly held bond. Data shows that the share of retirement wealth invested in directly held bonds is decreasing in the investor's investment horizon (Investment Company Institute (2018)). This finding amplifies the duration puzzle.

The shape of the equity glide-path in Figure 2(a) is not just a feature of the average TDF series. In fact, it is a feature of all TDF series in our dataset (see Figure 16 in Appendix B for all individual equity glide-paths). In all TDF series, the reduction in equity exposure over the investor's life-cycle leads to an increase in the fixed income share (see Figure 17 in Appendix B for all individual fixed income paths). The focus of this paper is interest rate risk management. To determine the interest rate sensitivity of the total portfolio, it is not sufficient to observe only the portfolio shares. We also need to observe the interest rate sensitivity of each asset class.

The most important source of interest rate risk exposure is the fixed income portfolio. The interest rate risk exposure of the fixed income portfolio can be measured by its duration. For most TDFs we observe the modified duration of the fixed income portfolio. We have already presented the average duration of fixed income assets over the life-cycle in Figure 1 in the introduction. A striking feature is that the modified duration on average hardly changes with age. This seems to be at odds with the theory.



(a) Portfolio Shares (b) Fixed Income Contribution to Duration

Figure 2. Portfolio shares and fixed income contribution to portfolio duration. Panel (a) illustrates the (weighted) mean shares of TDF assets invested in stocks, fixed income instruments (i.e., bonds), cash and other instruments as a function of age. This panel is based on 2019 data. Panel (b) shows the average contribution (in years) of fixed income instruments to the overall duration of TDF assets for both 2017 and 2019 data. This contribution is calculated as the fixed income portfolio share multiplied by the modified duration of the fixed income assets. The age on the horizontal axis is chosen such that the target date corresponds to age 65.

The contribution of fixed income assets to the overall TDF portfolio duration is equal to the fixed income portfolio share multiplied by its duration. Figure 2(b) shows the average fixed income contribution to overall portfolio duration over the life-cycle. We observe that there has been very little change between 2017 and 2019. On average, fixed income securities held by TDFs add little interest rate risk exposure. As a matter of fact, there is no TDF series, except one, in which the fixed income securities add more than 5 years to the overall portfolio duration for any target date (see Figure 18 in Appendix B for all individual fixed income contributions). The only exception is the *Dimensional Target Date Retirement Income Fund*. In the life-cycle strategy of Dimensional, fixed income securities add little duration for younger individuals. Once an individual enters her forties, however, the fixed income securities start to contribute significantly to overall portfolio duration, peaking at a contribution of 10 years around the age of 60.

Fixed income securities may not be the only source of exposure to interest rate risk. Equities can also provide exposure. Unfortunately it is not trivial to find a good measure of the interest rate sensitivity of stocks. The interest rate sensitivity will differ per stock portfolio. Moreover, our data does not allow us to estimate the difference in interest rate sensitivity of stock portfolios across TDFs. Given the lack of active fixed income duration management over the life-cycle, it is unlikely though that the interest rate sensitivity of stocks depends significantly on the investment horizon. If we assume that the interest rate duration of the stock portfolio equals 4.68, which is in line with, e.g., Bernanke and Kuttner (2005), the average overall duration of TDF portfolios is around 5 for all ages (see Figure 3).

As can be concluded from Figures 16, 17 and 18 (see Appendix B), we observe little cross-sectional and time variation in the portfolio strategies of TDFs. This observation is consistent with Bodie and Treussard (2007) who conclude that TDFs are only appropriate for investors with 'average' risk aversion. They are thus not appropriate for very risk-averse investors or investors who have a large equity exposure through their human capital. One of the reasons that may potentially explain the absence of cross-sectional and time variation in the portfolio strategies is inertia. As fund participants are passive with respect to moving assets from one fund to another fund, TDF managers have few incentives to offer a portfolio strategy that differs too much from the 'average' strategy in the market; see, e.g., Sandhya (2012).

The low overall duration and its flat life-cycle pattern seems puzzling from a theoretical point of view. Section 5 explores the question to what extent we can explain these findings using a rational life-cycle investment model. But first we introduce our benchmark model in Section 3. The optimal benchmark life-cycle policies are analyzed in Section 4.[4]

[4]We also briefly investigate the impact of non-time-separable preferences and owner-occupied housing and mortgage wealth on the duration of the optimal investment portfolio.

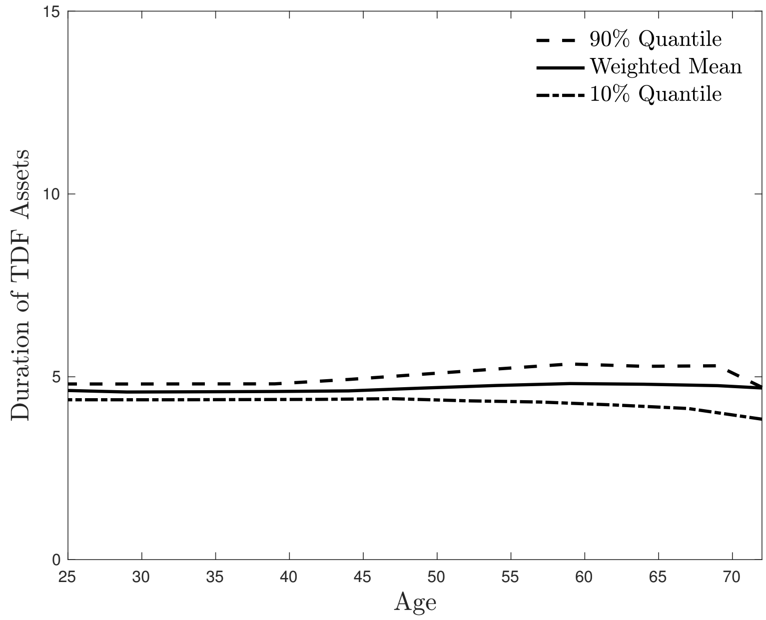


Figure 3. Duration of total TDF assets. The figure illustrates the (weighted) mean (solid line), the (weighted) 10% quantile (dash-dotted line) and the (weighted) 90% quantile (dashed line) of the duration of TDF assets as a function of age, assuming equities have a modified duration of 4.68 years and cash and 'other' assets have a duration of 0. The age on the horizontal axis is chosen such that the target date corresponds to age 65.

3 Benchmark Model

This section presents our benchmark model. The optimal benchmark life-cycle policies and the duration puzzle are introduced in Section 4.

3.1 Preferences

Time is continuous. Denote by t adult age, which corresponds to effective age minus 20. For ease of exposition, we assume that the adult age at which the individual dies is known in advance and is denoted by $T > 0$. Let $c(t)$ represent the individual's consumption choice at adult age t. The individual has CRRA preferences over consumption. Hence, the individual's expected lifetime utility is given by

$$U = \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}c(t)^{1-\gamma}\mathrm{d}t\right], \tag{3.1}$$

where $\delta \geq 0$ denotes the subjective rate of time preference, $\gamma > 0$ corresponds to the coefficient of relative risk aversion,[5] and $\mathbb{E}$ represents the (unconditional) expectation.

3.2 Asset Market and Wealth Accumulation

We consider a financial market with two state variables: the short-term interest rate $r(t)$ and the dividend payment $D(t)$. In Section 5.2, we add inflation risk to the model. The short-term interest rate and the dividend payment are modelled following Brennan and Xia (2002) and Benzoni et al. (2007), respectively. That is,

$$\mathrm{d}r(t) = \kappa_r \left(\bar{r} - r(t)\right) \mathrm{d}t + \sigma_r \mathrm{d}Z_r(t), \tag{3.2}$$

$$\mathrm{d}D(t) = \mu_D D(t)\, \mathrm{d}t + \sigma_D D(t)\, \mathrm{d}Z_D(t). \tag{3.3}$$

Here, $\bar{r}$ denotes the expected long-run short-term interest rate, $\kappa_r \geq 0$ is the mean reversion coefficient, μ_D models the expected growth in dividend payments, $Z(t) = (Z_r(t), Z_D(t))$ represents a vector of standard Brownian motions, and $\sigma = (\sigma_r, \sigma_D) \geq 0$ is a vector of diffusion coefficients.[6] We denote the correlation coefficient between $\mathrm{d}Z_r(t)$ and $\mathrm{d}Z_D(t)$ by ρ_{rD}.

The stochastic discount factor $M(t)$ satisfies

$$\frac{\mathrm{d}M(t)}{M(t)} = -r(t)\mathrm{d}t + \phi^\top \mathrm{d}Z(t), \tag{3.4}$$

where $\top$ denotes the transpose sign, and $\phi = (\phi_r, \phi_D)$ is a vector of factor loadings which determines the vector of market prices of risk associated with the underlying state variables. More specifically, we can obtain the market price of interest rate risk λ_r and the market price of dividend risk λ_D from ϕ_r and ϕ_D as follows:

$$\lambda_r = -\phi_r - \rho_{rD}\phi_D, \tag{3.5}$$

$$\lambda_D = -\phi_D - \rho_{rD}\phi_r. \tag{3.6}$$

[5] If $\gamma = 1$, then (3.1) reduces to

$$U = \mathbb{E}\left[\int_0^T e^{-\delta t} \log\left(c(t)\right) \mathrm{d}t\right].$$

[6] For notational convenience, we often write a (column) vector in the form $y = (y_1, y_2, \ldots, y_n)$, where y_i represents the ith element of y.

The individual invests her total wealth – which equals the sum of human capital and financial wealth – in a bond with fixed time to maturity h, a risky stock, and cash.

Let $P(t,h)$ denote the price at time t of a zero-coupon bond with time to maturity h. The bond price satisfies the following dynamics (see Appendix A.1):

$$\frac{\mathrm{d}P(t,h)}{P(t,h)} = \left(r(t) - \lambda_r \sigma_r B_r(h)\right)\mathrm{d}t - B_r(h)\,\sigma_r \mathrm{d}Z_r(t). \tag{3.7}$$

Here, $B_r(h) = \left(1 - e^{-\kappa_r h}\right)/\kappa_r \in [0,h]$ models the interest rate duration of the bond.[7] Note that $B_r(h)$ goes to h as interest rates become less predictable (i.e., as κ_r goes down).

We assume that the stock price is equal to the discounted value of all future dividends, in line with the dividend discount model (Gordon and Shapiro (1956) and Gordon (1959)). The stock price will now be subject to both changes in dividend payments ('cash-flow news') and changes in discount rates ('discount rate news').[8] Indeed, Appendix A.2 shows that we can write the relative change in the cum-dividend stock price $S(t)$ as follows:

$$\begin{aligned}\frac{\mathrm{d}S(t)}{S(t)} &= \left(r(t) - \lambda_r\sigma_r D_S(t) + \lambda_D\sigma_D\right)\mathrm{d}t + \sigma_D\,\mathrm{d}Z_D(t) - \left(D_S(t)\sigma_r + \sigma_D\rho_{rD}\right)\mathrm{d}Z_r(t)\\ &= \left(r(t) - \lambda_r\sigma_r D_S(t) + \lambda_D\sigma_D\right)\mathrm{d}t + \sigma_D\sqrt{1-\rho_{rD}^2}\,\mathrm{d}Z_U(t) - D_S(t)\sigma_r\,\mathrm{d}Z_r(t),\end{aligned} \tag{3.8}$$

where $Z_U(t)$ is a standard Brownian motion that is independent of $Z_r(t)$ and $D_S(t)$ denotes the interest rate duration of the stock, i.e.,

$$D_S(t) \equiv \frac{\int_t^\infty \widetilde{S}(t,s)B_r(s-t)\,\mathrm{d}s}{\int_t^\infty \widetilde{S}(t,s)\,\mathrm{d}s} - \frac{\sigma_D}{\sigma_r}\rho_{rD} \tag{3.9}$$

with $\widetilde{S}(t,s)$ given in Appendix A.2 (see (A16)).

We denote by $\omega(t) = (\omega_P(t), \omega_S(t))$ the vector of portfolio weights, with $\omega_P(t)$ the share of wealth invested in the bond at adult age t and $\omega_S(t)$ the share of wealth invested in the risky stock at adult age t. As a result, the share of wealth invested in cash at adult age t is given by $1 - \omega_P(t) - \omega_S(t)$. Let $W(t)$ denote the investor's total wealth at adult

[7] The term 'duration' can be defined in different ways. From this point onward, we will use the term 'duration' to refer to the change in relative value per percentage point change in $r(t)$. So, if $P(t,h)$ is the value at time t of a zero-coupon bond with time to maturity h, its duration at time t is equal to $\frac{\partial P(t,h)}{\partial r(t)}/P(t,h)$.

[8] Note that discount rate news is for a large part driven by changes in interest rates; see, e.g., Campbell (1987), Vuolteenaho (2002), Campbell and Vuolteenaho (2004), Cochrane (2011), and Campbell, Giglio, and Polk (2013).

age t which satisfies the following dynamic budget constraint:

$$\mathrm{d}W(t) = \left(r(t) + \omega(t)^\top \left[\mu(t) - r(t)\right]\right) W(t)\mathrm{d}t + \omega(t)^\top \Sigma(t) W(t)\mathrm{d}Z(t) - c(t)\mathrm{d}t, \tag{3.10}$$

with

$$\mu(t) = \begin{pmatrix} r(t) - \lambda_r \sigma_r B_r(h) \\ r(t) - \lambda_r \sigma_r D_S(t) + \lambda_D \sigma_D \end{pmatrix} \quad \text{and} \quad \Sigma(t) = \begin{pmatrix} -B_r(h)\,\sigma_r & 0 \\ -D_S(t)\sigma_r & \sigma_D \end{pmatrix}.$$

3.3 Maximization Problem

The individual faces the following dynamic maximization problem:

$$\begin{aligned} &\max_{c(t),\omega(t)} && \mathbb{E}\left[\int_0^T e^{-\delta t} \frac{1}{1-\gamma} c(t)^{1-\gamma}\,\mathrm{d}t\right] \\ &\text{s.t.} && \frac{\mathrm{d}W(t)}{W(t)} = \left(r(t) + \omega(t)^\top \left[\mu(t) - r(t)\right]\right)\mathrm{d}t + \omega(t)^\top \Sigma(t)\mathrm{d}Z(t) - \frac{c(t)}{W(t)}\mathrm{d}t. \end{aligned} \tag{3.11}$$

Section 4 analyzes and discusses the optimal benchmark policies over the individual's life-cycle.

3.4 Benchmark Parameter Values

Although we solve the benchmark model and its extensions (where possible) analytically, we will frequently illustrate the results for specific parameter values. Unless mentioned otherwise, the parameter values used will be as follows. We consider an individual who starts working at age 20, retires at age 65 (i.e., $T_R = 65-20 = 45$) and passes away at age 85 (i.e., $T = 85-20 = 65$). Her relative risk aversion parameter γ equals 5 and her time preference parameter is 3%.[9] We set the expected long-run short-term interest rate, i.e., $\bar{r}$, to 2%. The initial short-term interest rate $r(0)$ is assumed to be equal to its long-term mean (i.e., $r(0) = \bar{r}$). We set the interest rate volatility σ_r equal to 1%. The half-time of the interest rate η, which uniquely identifies the mean reversion coefficient κ_r, equals 20 years.[10] The interest rate risk premium λ_r is set equal to 7.5%. This implies for example that the expected excess return on a 30-year zero-coupon bond is 1.4 percent. We set

[9] We note that Samwick (1998) finds that time preference rates for U.S. households are between 3% and 4%.

[10] The half-time of the interest rate η is the time it takes for the interest rate to revert half way back to its long-term mean from its current level if no Brownian shocks arrive. The mean reversion coefficient κ_r can be computed from the half-time of the interest rate as follows: $\kappa_r = \log(2)/\eta$.

the interest rate duration of the stock $D_S(t)$ equal to 4.68 for all t which corresponds to the value found by Bernanke and Kuttner (2005). The dividend payment volatility σ_D and the market price of dividend risk λ_D are assumed to be equal to 18% and 0.2, respectively. Our parameter values imply an equity risk premium of about 4% which is in line with, e.g., Gomes, Kotlikoff, and Viceira (2008). Finally, we assume that the Brownian increments $\mathrm{d}Z(t) = (\mathrm{d}Z_r(t), \mathrm{d}Z_D(t))$ are uncorrelated (i.e., $\rho_{rD} = 0$). Where relevant we will highlight and discuss the impact of a change in the parameter values on the optimal life-cycle policies.

4 Optimal Benchmark Life-Cycle Policies

4.1 Optimal Benchmark Consumption Strategy

First, we determine the optimal benchmark consumption choice $c^*(t)$ using the martingale approach (Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989), and Cox and Huang (1991)). The individual's optimal consumption choice is given by (see Appendix A.3 for a proof)

$$c^*(t) = c^*(0) \exp\left\{\frac{1}{\gamma}\int_0^t \left(r(s) + \frac{1}{2}\phi^\top \rho\phi - \delta\right) \mathrm{d}s - \frac{1}{\gamma}\phi^\top \int_0^t \mathrm{d}Z(s)\right\}. \tag{4.1}$$

Here, $c^*(0)$ denotes the individual's optimal consumption choice at the beginning of the life-cycle which is chosen such that the market-consistent value of the optimal consumption stream is equal to the individual's total wealth.

4.2 Optimal Benchmark Portfolio Strategy

The vector of optimal benchmark portfolio weights $\omega^*(t)$ can be found using replication arguments. More specifically, the vector of optimal benchmark portfolio weights $\omega^*(t)$ is chosen such that fluctuations in total wealth exactly match fluctuations in the market-consistent value of optimal consumption. We find (see Appendix A.3 for a proof):

$$\omega_P^*(t) = \frac{1}{\gamma}\frac{\phi_r}{B_r\,(h)\,\sigma_r} + \frac{D_A(t)}{B_r\,(h)} - \omega_S^*(t)\frac{D_S(t)}{B_r(h)}, \tag{4.2}$$

$$\omega_S^*(t) = -\frac{1}{\gamma}\frac{\phi_D}{\sigma_D}. \tag{4.3}$$

Here, $D_A(t)$ denotes the interest rate sensitivity, or duration, of $A^*(t) \equiv W^*(t)/c^*(t)$, which we will refer to as the optimal annuity factor or the optimal inverse consumption-to-wealth ratio ($W^*(t)$ denotes the optimal wealth level implied by the optimal consumption choices). So, $D_A(t)$ measures how sensitive the speed of wealth drawdown is to interest rate fluctuations. We can write $D_A(t)$ as follows:

$$D_A(t) = \left(1 - \frac{1}{\gamma}\right) \int_0^{T-t} \frac{V^*(t,h)}{V^*(t)} B_r(h) \,\mathrm{d}h, \tag{4.4}$$

where $V^*(t,h)$ is the market-consistent value at time t of the (stochastic) optimal consumption choice at time $t+h$ and $V^*(t) \equiv \int_0^{T-t} V^*(t,h)\,\mathrm{d}h$.

We denote by $\omega_P^*(t)$ the optimal share of total wealth invested in a bond with fixed time to maturity h. There are two reasons why the individual prefers to allocate part of her total wealth to a bond. The first reason (represented by the first term on the right-hand side of (4.2)) is a speculative one: the individual wants to profit from the interest rate risk premium $-\lambda_r\sigma_r B_r(h) \geq 0$.

The second reason (represented by the second term on the right-hand side of (4.2)) is the 'Mertonian' hedging demand. The individual wants to hedge against unfavorable developments in the investment opportunity set, i.e., a decline in the interest rate. This hedging demand depends on the duration of the optimal annuity factor $D_A(t)$. The larger this duration is, the bigger the share of total wealth allocated to the bond will be (assuming fixed $B_r(h)$).

Note that the third term on the right-hand side of (4.2) reduces the optimal bond portfolio weight. Intuitively, because the stock already provides a partial hedge against interest rate risk, the investor has less need to invest in the bond.

Figure 4 shows the median duration of the optimal annuity factor over the life-cycle for various coefficients of relative risk aversion and various half-times of the interest rate. As shown by this figure, the duration declines as the individual becomes older. Indeed, the younger the individual, the longer her investment period, and hence the more sensitive the optimal annuity factor is to changes in the interest rate. Furthermore, the optimal duration is an increasing function of the relative risk aversion coefficient. We can decompose the effect of an interest rate shock on the optimal inverse consumption-to-wealth ratio in an income and a substitution effect. While the relative risk aversion coefficient does not affect the income effect, the substitution effect increases as the relative risk aversion coefficient decreases. As is well-known, the substitution effect exactly cancels against the income effect when the relative risk aversion coefficient is one (log utility).

In that case, the current optimal inverse consumption-to-wealth ratio does not depend on the interest rate (i.e., $D_A(t) = 0$) and the hedging demand is zero. Intuitively, the gain from a rise in interest rates is fully allocated towards future consumption, so that the current optimal inverse consumption-to-wealth ratio is not affected. Furthermore, we note that the duration of the optimal inverse consumption-to-wealth ratio goes down as the half-time of the interest rate decreases. Indeed, a larger degree of predictability of future interest rates implies less interest rate risk, and hence a lower sensitivity of the optimal inverse consumption-to-wealth ratio to an unexpected shock in the interest rate.

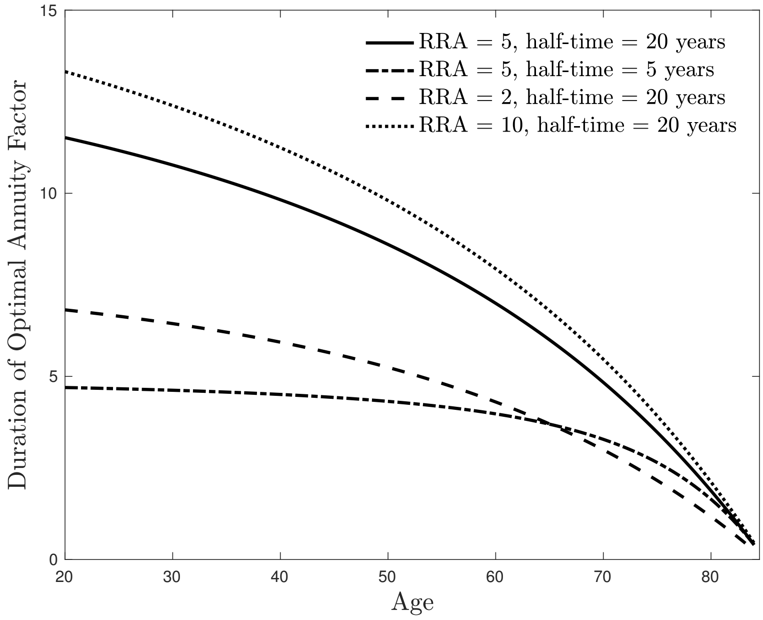
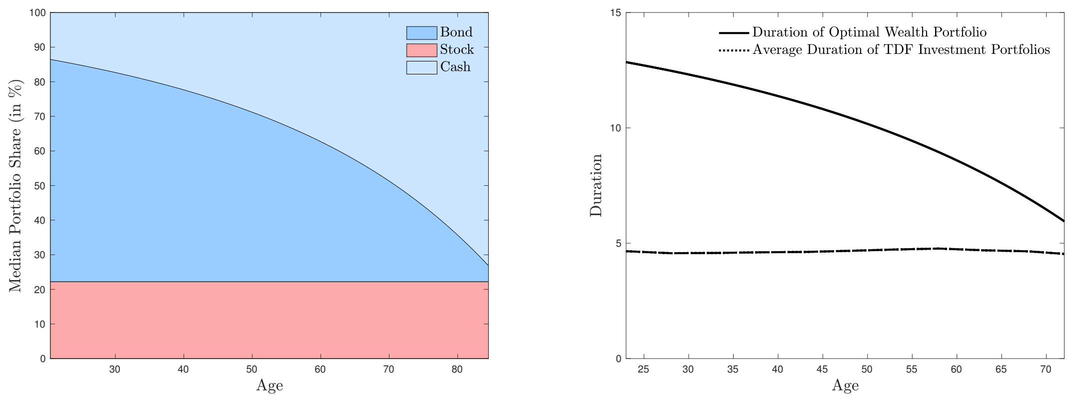


Figure 4. Illustration of the duration of the optimal annuity factor. The figure illustrates the median duration of the optimal annuity factor over the life-cycle for various coefficients of relative risk aversion and various half-times of the interest rate. The benchmark parameter values are given in Section 3.4.

We denote by $\omega_S^*(t)$ the optimal share of total wealth invested in the risky stock. The individual invests part of her total wealth in the risky stock so as to pick up the equity risk premium $-\lambda_r \sigma_r D_S(t) + \lambda_D \sigma_D \geq 0$. As is well-known since Merton (1969), under constant relative risk aversion, the optimal share of total wealth invested in the risky stock does not change over the individual's life-cycle.

4.3 The Duration Puzzle

Figure 5(a) illustrates the median portfolio shares over the life-cycle. We assume that the individual invests her total wealth in a 30-year zero-coupon bond, a risky stock, and cash. As shown by this figure, the demand for the bond decreases as the individual becomes older. A bond – which hedges against interest rate risk – is especially valuable for a young individual. Indeed, a young individual features a long investment horizon, so that an interest rate shock yields a large impact on the value of optimal consumption. The optimal life-cycle pattern of the bond portfolio weight is *not* consistent with empirical evidence: long-term bonds are more or less absent in TDF portfolios of young individuals; see Figure 2(a). As a result, as shown by Figure 5(b), the average duration level of TDF investment portfolios is too low compared to the median duration level of the optimal investment portfolio. Furthermore, we find that the optimal duration is *decreasing* in the investment horizon, while the average observed duration does not depend on the investment horizon. We call these stylized facts the *duration puzzle*.[11]



(a) Portfolio Shares

(b) Comparison of Durations

Figure 5. Illustration of the duration puzzle. Panel (a) illustrates the median shares of total wealth invested in a 30-year bond, a risky stock and cash as a function of age. Panel (b) shows the median duration of the optimal wealth portfolio (solid line) and the average duration of TDF investment portfolios (dash-dotted line). The benchmark parameter values are given in Section 3.4.

The next sections explore the impact of human capital, inflation risk, portfolio restrictions, non-time-separable preferences and owner-occupied housing and mortgage

[11] We note that if $1/\gamma$ is equal to unity, investors hold bonds only for speculative reasons. Hence, this may provide an explanation for the duration puzzle. However, empirical estimates of $1/\gamma$ are substantially lower than one; see, e.g., Hall (1988), Campbell (2003), Yogo (2004), and Best, Cloyne, Ilzetzki, and Kleven (2018).

wealth on the duration of the optimal investment portfolio. We address the question whether these model extensions are able to explain the observed duration pattern as well as the observed duration level.

It can be costly to implement a suboptimal investment strategy. To illustrate this, we calculate what the welfare loss will be if an investor of a particular age holds an investment portfolio with a suboptimal duration during the first year. After the first year, she holds the optimal investment portfolio for the rest of her life; see equations (4.2) and (4.3). The suboptimal duration corresponds to the average observed duration of total TDF assets; see the dash-dotted line in Figure 5(b). Figure 6 reports our results. We observe that welfare losses can be enormous, especially for young investors.

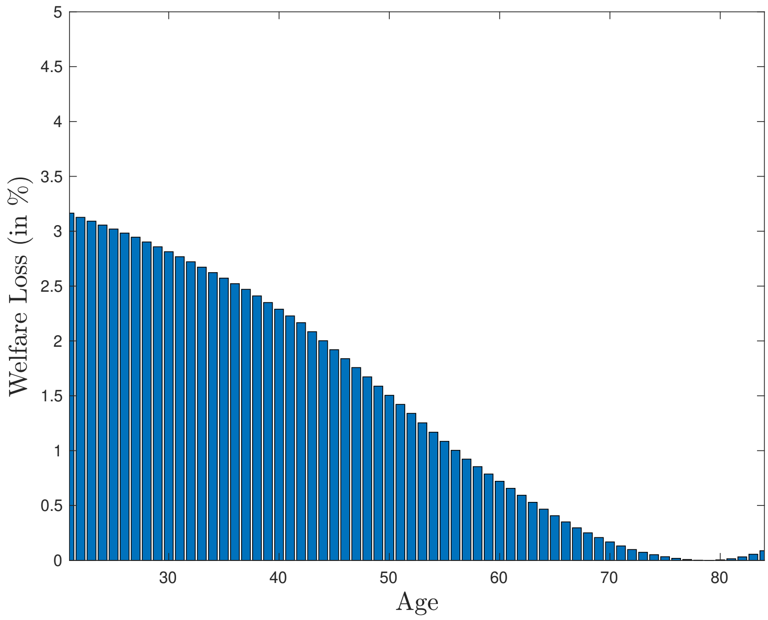


Figure 6. Welfare costs. This figure shows what the welfare loss will be if an investor of a particular age holds an investment portfolio with a suboptimal duration during the first year. After the first year, she holds the optimal investment portfolio for the rest of her life. The suboptimal duration corresponds to the average observed duration of total TDF assets. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. The benchmark parameter values are given in Section 3.4.

We note that the shape of the life-cycle pattern of the bond portfolio weight is robust to changes in the parameters. Indeed, as can be observed from (4.2), a change in γ, ϕ_r or σ_r leads to a parallel shift in the life-cycle pattern of the bond portfolio weight but it leaves its shape unaffected.[12] Moreover, we observe from (4.2) that a change in κ_r

[12]A change in γ, ϕ_r or σ_r also leads to change in the duration of the optimal annuity factor $D_A(t)$. However, this effect is rather limited.

causes both the speculative and the hedging bond portfolio weight to change. However, its impact on the shape of the life-cycle pattern of the bond portfolio weight is relatively small (second order). As a consequence of the just mentioned findings, the duration puzzle will hold true irrespective of the parameter values.

Finally, we note that our model assumes constant bond risk premiums. This assumption does not impact our conclusions. In fact, the presence of time-varying bond risk premiums strengthens the duration puzzle. Indeed, long-term investors now have an additional reason to invest in bonds: to hedge against time variation in bond risk premiums.

5 Alternative Model Specifications

5.1 Human Capital

One potential explanation for the low duration of TDF investment portfolios could be that our benchmark model ignores human capital. This section explores whether the presence of human capital could justify the low durations we observe in the data. We define human capital as the discounted value of future earnings, which consists of labor income and social security. Like Bodie et al. (1992), we assume that future earnings are similar to a traded asset. In particular, we assume that labor income and social security are risk-free and do not vary with age.[13] The assumption that outside income is risk-free is made not only to simplify the analysis. Since we try to explain the duration puzzle, this somewhat extreme case should be most promising, as it maximizes the duration of outside income. Given this assumption, we can interpret human capital as a bond. Let $D_H(t)$ denote the duration at time t of human capital which is defined as follows:

$$D_H(t) = \int_0^{T-t} \frac{H(t,h)}{H(t)} B_r(h)\, \mathrm{d}h, \tag{5.1}$$

where $H(t,h)$ is the time-t value of income received at time $t+h$ and $H(t) \equiv \int_0^{T-t} H(t,h)\, \mathrm{d}h$.

The individual can invest only her financial wealth – which equals total wealth minus human capital – in the financial market. When the investor makes the asset allocation

[13] We have also computed our results using the realistically calibrated income profile of Cocco et al. (2005). Our results remain qualitatively unchanged. In particular, long-term bonds are still an important asset for individuals aged between 35 and 65.

decision, she takes into account that she already possess a long-lived asset (i.e., human capital). Denote by $\widehat{\omega}(t) = (\widehat{\omega}_P(t), \widehat{\omega}_S(t))$ a vector consisting of the shares of *financial* wealth invested in the risky assets. We find that the optimal portfolio weights $\widehat{\omega}^*_P(t)$ and $\widehat{\omega}^*_S(t)$ are given by (see Appendix A.4 for a proof)

$$\widehat{\omega}^*_P(t) = \frac{W(t)}{F(t)}\omega^*_P(t) - \frac{H(t)}{F(t)}\frac{D_H(t)}{B_r(h)}, \tag{5.2}$$

$$\widehat{\omega}^*_S(t) = \frac{W(t)}{F(t)}\omega^*_S(t), \tag{5.3}$$

where $F(t) = W(t) - H(t)$ denotes financial wealth at adult age t, and $\omega^*_P(t)$ and $\omega^*_S(t)$ are the optimal portfolio weights given earlier by (4.2) and (4.3), respectively.

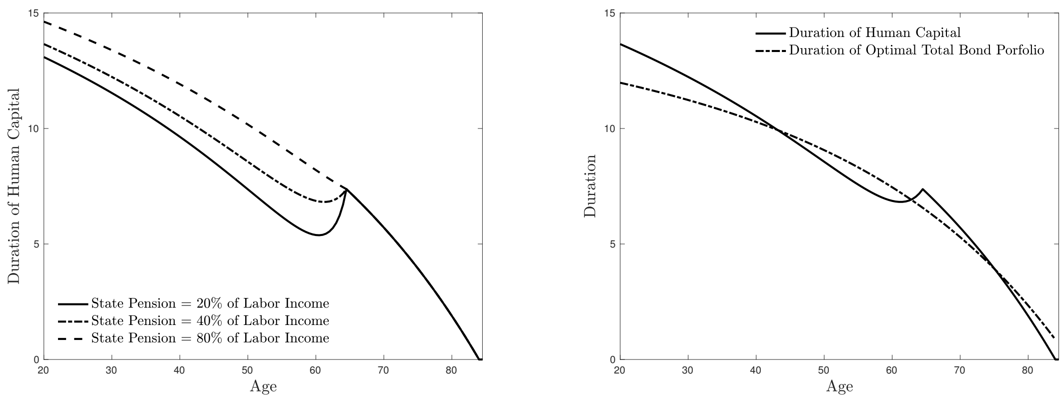
Consistent with conventional wisdom and as shown by Bodie et al. (1992), the share of financial wealth invested in the risky stock is not constant, but decreases on average with age. Intuitively, because human capital is not exposed to stock market risk, the individual invests a large part of her financial wealth in the risky stock to obtain the preferred overall exposure to stock market risk. Because human capital becomes relatively less important as the individual ages, the share of financial wealth invested in the risky stock decreases on average over the individual's life-cycle.

The impact of human capital on the demand for the bond is less obvious. We can write the share of financial wealth invested in the bond as follows:

$$\widehat{\omega}^*_P(t) = \omega^*_P(t) + \frac{H(t)}{F(t)}\frac{D_W(t) - D_H(t)}{B_r(h)}. \tag{5.4}$$

Here, $D_P(t) \equiv \omega^*_P(t)B_r(h) = \phi_r/(\gamma\sigma_r) + D_A(t) - \omega^*_S(t)D_S(t)$ denotes the duration of the optimal total bond portfolio, with $D_A(t)$ representing the duration of the optimal annuity factor (see (4.4)).

The main driver for the individual to invest in a bond is interest rate hedging. The individual already owns a long-lived asset (i.e., human capital) which provides a hedge against interest rate risk. However, the duration of human capital $D_H(t)$ is typically not equal to the duration of the optimal total bond portfolio $D_P(t)$. If $D_H(t)$ is smaller than $D_P(t)$, then human capital exhibits insufficient exposure to interest rate risk. As a result, the individual should increase bond investments to obtain an adequate hedge against interest rate risk. Conversely, if $D_H(t)$ is larger than $D_P(t)$, then human capital provides too much exposure to interest rate risk. In that case, the individual should reduce bond investments to achieve the preferred hedge against interest rate risk. Note that the duration of human capital $D_H(t)$ may indeed exceed the duration of the optimal total bond portfolio $D_P(t)$. This may in particular be the case if the coefficient of relative risk aversion is relatively low and the size of the state pension is sufficiently large. Figure 7 shows the median duration of human capital over the life-cycle for various levels of state pension. The figure also compares the median duration of human capital with the median duration of the optimal total bond portfolio.



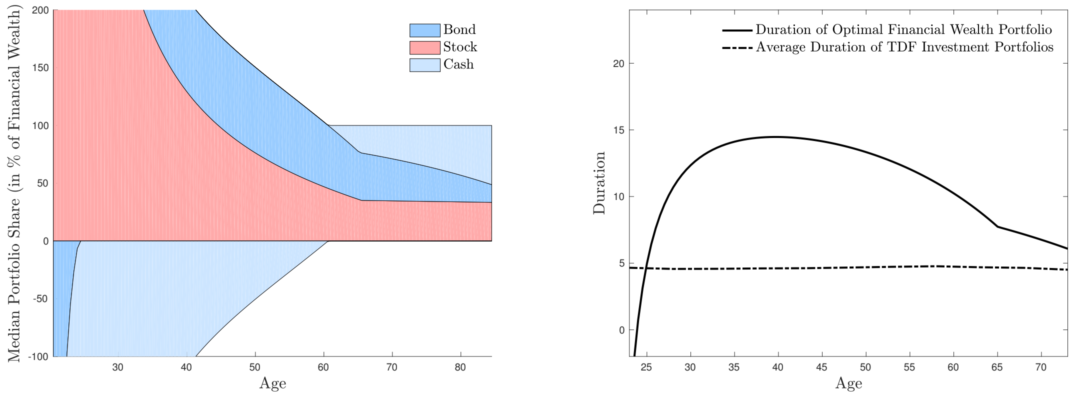
(a) Impact of State Pension on Duration

(b) Comparison of Durations

Figure 7. Illustration of the duration of human capital. Panel (a) illustrates the median duration of human capital over the life-cycle for various levels of state pension. Panel (b) compares the median duration of human capital with the median duration of the optimal total bond portfolio. This panel assumes that the state pension is equal to 40% of labor income. The benchmark parameter values are given in Section 3.4.

Figure 8(a) shows the median shares of financial wealth invested in a 30-year bond, a risky stock, and cash. As shown by this figure, a substantial part of the investment portfolio consists of the bond, especially for individuals aged below 65 years. Although human capital causes the demand for the bond to decrease at young ages, it remains an important asset for individuals aged between 35 and 65. We note that the demand for cash is negative for most ages: the individual borrows money to invest in the financial market. Figure 8(b) illustrates the median duration of the optimal financial wealth portfolio and the average duration of TDF investment portfolios. Although human capital may to some extent rationalize the low duration of the TDF investment portfolios of young individuals, it still remains difficult to explain the duration puzzle.

Figure 9 illustrates what the welfare loss will be if an investor of a particular age holds an investment portfolio with a suboptimal duration during the first year. As in Figure 6, the suboptimal duration corresponds to the observed average duration of total TDF



(a) Portfolio Shares

(b) Comparison of Durations

Figure 8. Impact of human capital on portfolio strategy and duration. Panel (a) illustrates the median shares of financial wealth invested in a 30-year bond, a risky stock and cash as a function of age. Panel (b) shows the median duration of the optimal financial wealth portfolio (solid line) and the average duration of TDF investment portfolios (dash-dotted line). The figure assumes that the state pension equals 40% of labor income. The benchmark parameter values are given in Section 3.4.

assets. What is different is that the optimal investment strategy now takes risk-free labor income into account. Although the welfare losses are lower compared to Figure 6, they are still substantial, especially for individuals aged between 35 and 75.

Figure 10 explores how sensitive the median duration of the financial wealth portfolio is to the level of the state pension and the coefficient of relative risk aversion. We conclude that the median duration of the optimal financial wealth portfolio exhibits a hump-shaped pattern for a wide range of parameter values. Furthermore, we observe that workers with a relatively high level of state pension have a lower demand for long-term bonds than workers with a relatively low level of state pension. Indeed, their human capital, which includes state pension, already provides a large hedge against interest rate risk. Also, investors with a low relative risk aversion coefficient invest a relatively small part of financial wealth in long-term bonds: they do not value a stable consumption stream as much as investors with a high relative risk aversion coefficient do. Finally, as shown by Figure 10, a relative risk aversion coefficient of 2 seems most promising in explaining the duration puzzle.

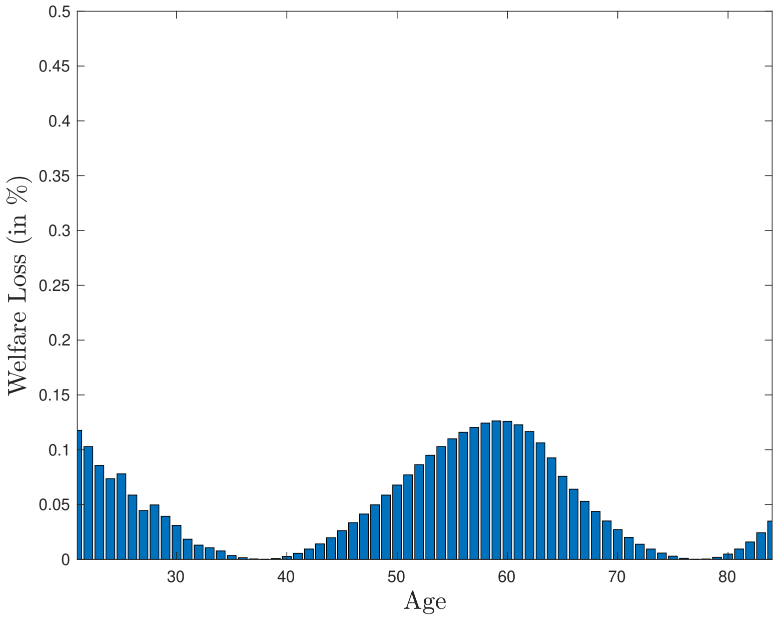
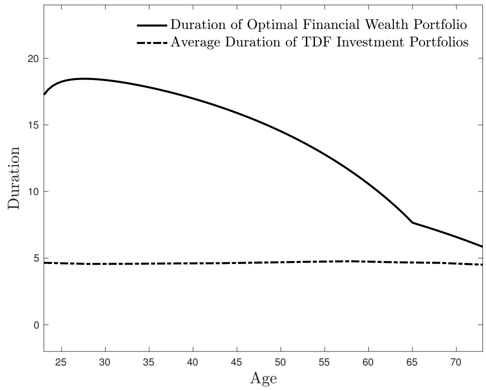
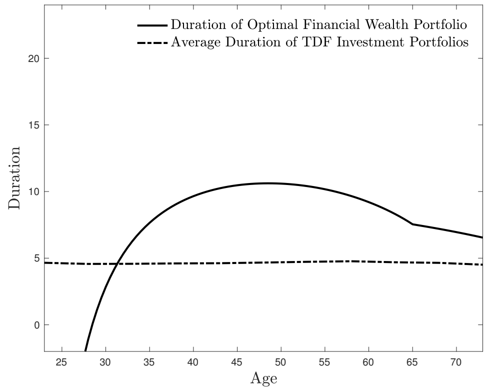


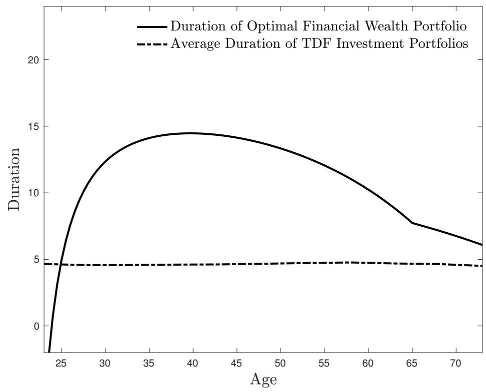
Figure 9. Welfare costs. This figure shows what the welfare loss will be if an investor of a particular age holds an investment portfolio with a suboptimal duration during the first year. After the first year, she holds the optimal investment portfolio for the rest of her life. The optimal investment strategy takes risk-free labor income into account and assumes that the state pension equals 40% of labor income. The suboptimal duration corresponds to the observed average duration of total TDF assets. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. The benchmark parameter values are given in Section 3.4.



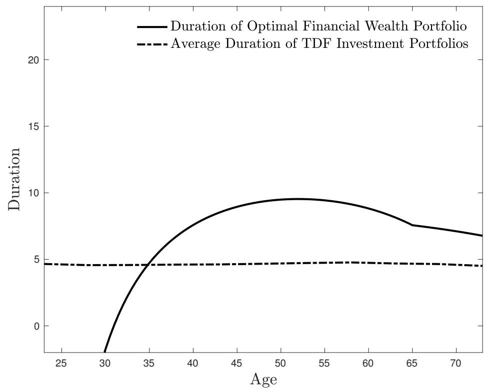
(a) $s = 20\%$ of labor income, $\gamma = 5$



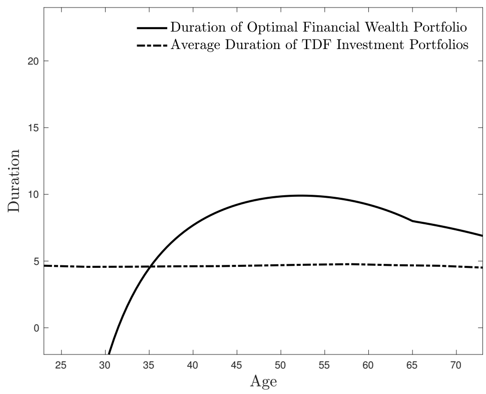
(b) $s = 20\%$ of labor income, $\gamma = 2$



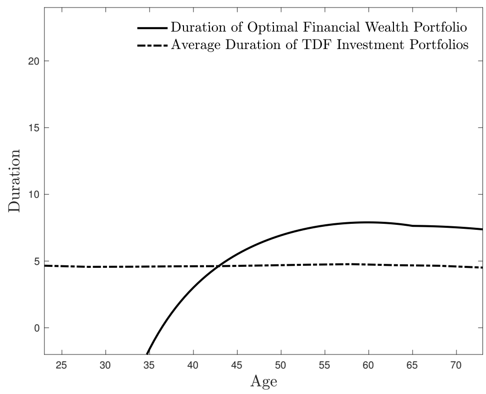
(c) $s = 40\%$ of labor income, $\gamma = 5$



(d) $s = 40\%$ of labor income, $\gamma = 2$



(e) $s = 80\%$ of labor income, $\gamma = 5$



(f) $s = 80\%$ of labor income, $\gamma = 2$

Figure 10. Sensitivity of duration to level of state pension and relative risk aversion coefficient. The figure illustrates the median duration of the optimal financial wealth portfolio for various levels of state pension and various relative risk aversion coefficients. The symbol s denotes the social security payment. The benchmark parameter values are given in Section 3.4.

5.2 Inflation Risk

So far, we have ignored the role of inflation risk. Both Campbell and Viceira (2002) and Brennan and Xia (2002) point out that nominal bonds are less desirable to hedge interest rate risk in the presence of (persistent) inflation shocks. Therefore, this section analyzes the impact of adding inflation risk to the model on the duration of the optimal investment portfolio. We now interpret $r(t)$ as the *real* interest rate. Just like the real interest rate $r(t)$, we assume that the rate of inflation $\pi(t)$ follows an Ornstein-Uhlenbeck process:

$$\mathrm{d}\pi(t) = \kappa_\pi \left(\bar{\pi} - \pi(t)\right) \mathrm{d}t + \sigma_\pi \mathrm{d}Z_\pi(t), \tag{5.5}$$

where $\bar{\pi}$ denotes the expected long-run inflation rate, $\kappa_\pi \geq 0$ is the mean reversion coefficient, $Z_\pi(t)$ represents a Brownian motion, and $\sigma_\pi \geq 0$ is a diffusion coefficient. We denote the market price of inflation risk by λ_π.

The stochastic discount factor $M(t)$ now satisfies:

$$\frac{\mathrm{d}M(t)}{M(t)} = -r(t)\mathrm{d}t + \phi^\top \mathrm{d}Z(t), \tag{5.6}$$

with $\phi = (\phi_r, \phi_D, \phi_\pi)$ representing a vector of factor loadings, and $Z(t) = (Z_r(t), Z_D(t), Z_\pi(t))$. The coefficients $\rho_{r\pi}$ and $\rho_{D\pi}$ model the correlation between $\mathrm{d}Z_r(t)$ and $\mathrm{d}Z_\pi(t)$ and the correlation between $\mathrm{d}Z_D(t)$ and $\mathrm{d}Z_\pi(t)$, respectively.

5.2.1 Complete Market for Inflation Risk Hedging

Let us first assume that the market for inflation risk hedging is complete. In our setup, this implies that the investor must have access to cash and three linearly independent assets. We achieve this by assuming that the investor has the opportunity to invest in an inflation-linked bond with fixed time to maturity h, a risky stock, a nominal bond with fixed time to maturity h_N, and cash. Let $p(t, h_N)$ be the *real* price at adult age t of a nominal zero-coupon bond with time to maturity h_N. It satisfies the following dynamics (see Appendix A.1):

$$\begin{aligned}\frac{\mathrm{d}p(t, h_N)}{p(t, h_N)} &= \left(r(t) - \lambda_r \sigma_r B_r(h_N) - \lambda_\pi \sigma_\pi B_\pi(h_N)\right) \mathrm{d}t \\ &\quad - B_r(h_N)\, \sigma_r \mathrm{d}Z_r(t) - B_\pi(h_N)\, \sigma_\pi \mathrm{d}Z_\pi(t),\end{aligned} \tag{5.7}$$

with $B_\pi(h_N) = \left(1 - e^{-\kappa_\pi h_N}\right)/\kappa_\pi \in [0, h_N]$ representing the inflation rate sensitivity of the nominal bond.

Denote by $\widehat{\omega}_p(t)$ the share of financial wealth invested in the nominal bond. We find that the optimal nominal bond share for an investor who maximizes CRRA utility (see (3.1)) is given by (see Appendix A.3)

$$\widehat{\omega}_p^*(t) = \frac{W(t)}{F(t)} \frac{1}{\gamma} \frac{\phi_\pi}{B_\pi(h_N)\,\sigma_\pi}. \tag{5.8}$$

The investor allocates part of her financial wealth to a nominal bond for a speculative reason: the nominal bond allows her to pick up the inflation risk premium $-\lambda_\pi \sigma_\pi B_\pi(h_N) \geq 0$. The size of the speculative demand is positively related to ϕ_π / σ_π and negatively related to the individual's coefficient of relative risk aversion γ. Because these parameters are assumed to be constant and we keep the time to maturity h_N fixed, the optimal share of *total* wealth invested in the nominal bond $\omega_p^*(t) = F(t)/W(t) \cdot \widehat{\omega}_p^*(t)$ does not change over the investor's life-cycle.

The optimal share of financial wealth invested in the stock $\widehat{\omega}_S^*(t)$ does not change as a result of adding inflation risk to the model. However, the presence of inflation risk does have an impact on the optimal share of financial wealth invested in the inflation-linked bond $\widehat{\omega}_P^*(t)$. We find (see Appendix A.3 for the derivation)

$$\widehat{\omega}_P^*(t) = \omega_P^*(t) \frac{W(t)}{F(t)} - \frac{H(t)}{F(t)} \frac{D_H(t)}{B_r(h)} - \widehat{\omega}_p^*(t) \frac{B_r(h_N)}{B_r(h)}, \tag{5.9}$$

where $\omega_P^*(t)$ is again the optimal bond portfolio weight in absence of outside income and inflation risk (see (4.2)). Comparing (5.2) with (5.9), we observe that in a setting with hedgeable inflation risk, the demand for the inflation-linked bond is lower. Indeed, the nominal bond already provides a partial hedge against real interest rate risk. The duration of optimal total wealth is not affected by inflation risk. It remains equal to $\phi_r / (\gamma \sigma_r) + D_A(t)$. The duration of optimal financial wealth is also not affected by inflation risk. Hence, in case of a complete market for inflation risk hedging, the presence of inflation risk does not help us to explain the duration puzzle.

5.2.2 Incomplete Market for Inflation Risk Hedging

The market for inflation-linked bonds is typically very small and can be considered as a missing market. Therefore, we now assume that the inflation-linked bond does not exist. The investor thus has the opportunity to invest in only two risky assets: a risky stock and a nominal bond with time to maturity h_N. In that case, a closed-form solution does not exist; see Liu (2007). We can find a solution in closed-form however if we consider a terminal wealth problem. It turns out that the intuition caries over to the intertemporal consumption problem. We therefore first present the solution to the terminal wealth problem in closed-form and then solve the intertemporal consumption problem numerically.

Appendix A.5 derives the optimal portfolio weights in closed-form for the case where the investor maximizes utility of terminal wealth $u\left(W\left(T_R\right)\right)$ (and where human capital is absent). Let us assume for the sake of simplicity that the Brownian increments $(\mathrm{d}Z_r(t), \mathrm{d}Z_D(t), \mathrm{d}Z_\pi(t))$ are uncorrelated. Furthermore, we assume that the duration of the stock, i.e., $D_S(t)$, is equal to zero. We now find the following closed-form solution:[14]

$$\omega_S^*(t) = -\frac{1}{\gamma}\frac{\phi_D}{\sigma_D}, \tag{5.10}$$

$$\omega_p^*(t) = \frac{1}{\gamma}\frac{\phi_r B_r\left(h_N\right)\sigma_r + \phi_\pi B_\pi\left(h_N\right)\sigma_\pi}{B_r^2\left(h_N\right)\sigma_r^2 + B_\pi^2\left(h_N\right)\sigma_\pi^2} + \left(1-\frac{1}{\gamma}\right)\frac{B_r\left(T_R-t\right)}{B_r\left(h_N\right)}\frac{1}{1+b^2}, \tag{5.11}$$

with $b = B_\pi\left(h_N\right)\sigma_\pi / \left(B_r\left(h_N\right)\sigma_r\right)$ representing the ratio between the exposure of the nominal bond to inflation innovations $\mathrm{d}Z_\pi(t)$ and the exposure of the nominal bond to real interest rate innovations $\mathrm{d}Z_r(t)$. In the absence of correlation between the Brownian increments, the demand for the risky stock is not affected. The demand for the nominal bond shows an intuitive pattern. First, the investor invests her wealth in the bond for speculative reasons. While in a complete market the inflation risk premium and the interest rate risk premium can be disentangled, this is no longer the case in an incomplete market setting. Therefore, the investor considers these risk premiums jointly. The idea that the speculative demand is proportional to the ratio between the risk premium and the variance of an asset is preserved though.

We observe something similar when considering the hedging demand. The only difference between the hedging demand in the complete market case and the hedging demand in the incomplete market case is the additional term $1/\left(1+b^2\right)$. Without inflation risk (i.e., $\sigma_\pi = 0$, so that $b = 0$), this term is unity and we are back in the complete market case. If $\sigma_\pi > 0$, this term causes the hedging demand to decrease. This is in line with the intuition that inflation risk makes the nominal bond a suboptimal hedging instrument. The coefficient $b = B_\pi\left(h_N\right)\sigma_\pi / \left(B_r\left(h_N\right)\sigma_r\right)$ determines by how much the hedging demand is reduced.

When we compare the duration of optimal total wealth at time t in the complete

[14] Appendix A.5 presents the solution for the case with correlated Brownian increments.

market case, i.e., $D_C(T_R - t)$, with the duration of optimal total wealth at time t in the incomplete market case, i.e., $D_I(T_R - t)$, we observe the following:

$$D_I(T_R - t) = D_C(T_R - t)\frac{1}{1+b^2} + \frac{1}{\gamma}\frac{\phi_\pi}{\sigma_r}\frac{b}{1+b^2}, \tag{5.12}$$

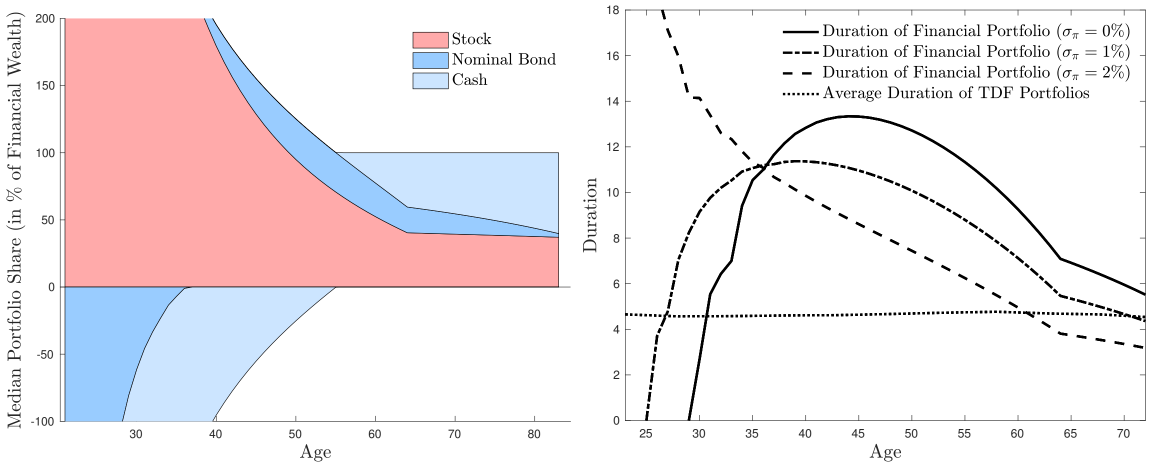
with

$$D_C(T_R - t) = \frac{1}{\gamma}\frac{\phi_r}{\sigma_r} + \left(1 - \frac{1}{\gamma}\right) B_r\left(T_R - t\right). \tag{5.13}$$

If $\phi_\pi = 0$, we observe from (5.12) that the duration of optimal total wealth in the incomplete market setting is lower by a factor $1/\left(1+b^2\right)$. If $\phi_\pi > 0$, the speculative demand for inflation risk creates a second effect in the other direction. Unless the inflation risk premium is very high, the first effect dominates the second effect and the duration is lowered by the absence of a complete market.

We now compute the optimal portfolio weights for the case where the investor maximizes expected lifetime utility (3.1). We use the method of numerical backward induction to arrive at the optimal portfolio weights; Appendix C provides more details on the numerical solution technique. Figure 11(a) shows the median shares of financial wealth invested in a risky stock, a 30-year nominal bond, and cash. Comparing Figure 8(a) with Figure 11(a), we observe that, consistent with our finding in the terminal wealth problem, the presence of unhedgeable inflation risk reduces the demand for the long-term bond. Figure 11(b) compares the duration of the optimal financial wealth portfolio with the average duration of TDF investment portfolios. We conclude that, while the presence of unhedgeable inflation risk does help to create a lower duration of the optimal financial wealth portfolio, it does not help us to justify the observed life-cycle pattern of the duration of the investment portfolio (unless σ_π is assumed to be relatively high).

Figure 12 shows the welfare losses associated with implementing a suboptimal duration. This figure should be interpreted in a similar fashion as Figures 6 and 9. The optimal investment strategy takes risk-free labor income into account. Furthermore, the financial market does not include inflation-linked bonds. Compared to Figures 6 and 9, the welfare losses are lower. However, middle-aged individuals still incur a substantial welfare loss by not adequately hedging interest rate risk.



(a) Portfolio Shares

(b) Comparison of Durations

Figure 11. Impact of unhedgeable inflation risk on portfolio strategy and duration. Panel (a) illustrates the median shares of financial wealth invested in a risky stock, a 30-year nominal bond and cash as a function of age assuming $\sigma_\pi = 1\%$. Panel (b) shows the median duration of the optimal financial wealth portfolio for various values of σ_π and the average duration of TDF investment portfolios (dash-dotted line). The figure assumes that the state pension equals 40% of labor income. Furthermore, we assume no correlation between the Brownian increments, a half-time of the inflation rate of 10 years, and an inflation risk premium ϕ_π of zero. The benchmark parameter values are given in Section 3.4.

5.3 Portfolio Restrictions

Section 5.1 showed that the financial portfolio becomes highly leveraged whenever human capital makes up a big part of total wealth. One may worry that such levels of financial leverage are hard to obtain in practice as human capital can not be used as collateral. Therefore, we now explore the impact of portfolio restrictions on the optimal portfolio policies. More specifically, we add the restriction that all portfolio weights have to lie between zero and one.[15] This restriction implies that we can no longer derive the optimal portfolio policies in closed-form. Hence, we resort to numerical optimization. Appendix C outlines the numerical solution technique.

Figure 13(a) illustrates the median shares of financial wealth invested in a 30-year bond, a risky stock, and cash. We assume no inflation risk. For the median scenario, the portfolio constraints are binding all the way up to an age of 60. Above this age, the

[15]Notice that the maximum interest rate risk exposure is determined by the combination of the restriction on the vector of portfolio weights and the duration of the bond. We use a bond with a rather long duration, a 30-year zero-coupon bond, because the restriction on the vector of portfolio weights is already rather restrictive. The restriction rules out that the individual can borrow money against her own financial wealth. In practice, however, this is possible for example through derivative instruments such as swap contracts.

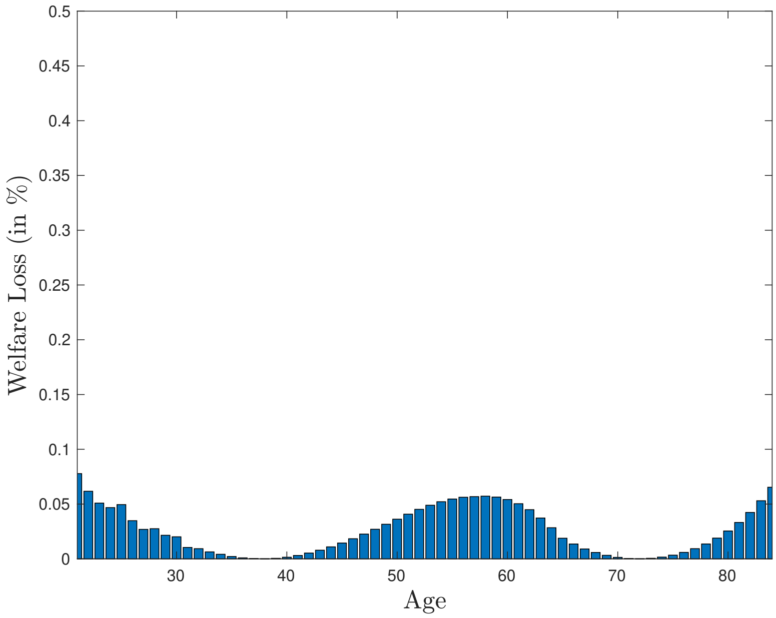
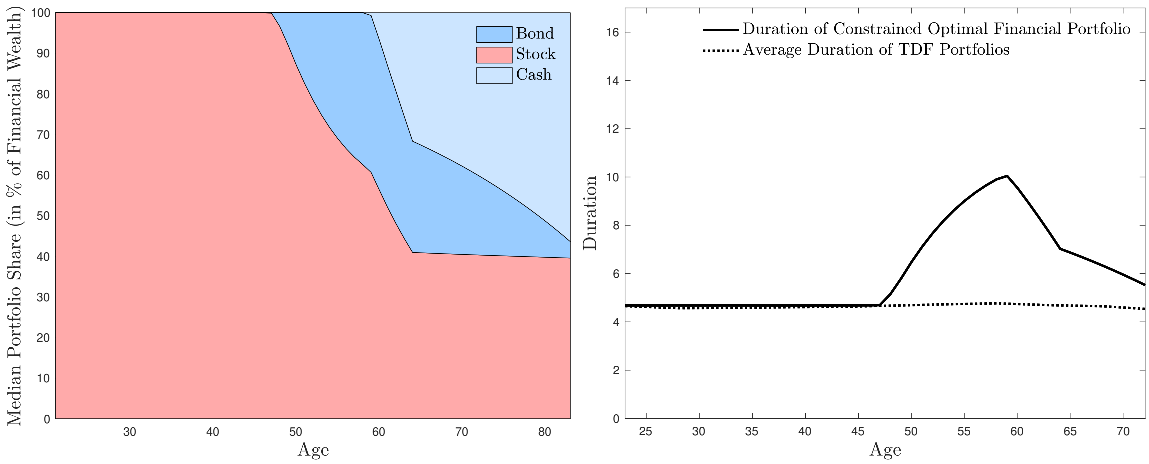


Figure 12. Welfare costs. This figure shows what the welfare loss will be if an investor of a particular age holds an investment portfolio with a suboptimal duration during the first year. After the first year, she holds the optimal investment portfolio for the rest of her life. The optimal investment strategy takes risk-free labor income into account and assumes that the state pension equals 40% of labor income. The financial market does not include inflation-linked bonds. Furthermore, we assume no correlation between the Brownian increments, a half-time of the inflation rate of 10 years, an inflation rate volatility σ_π of 1%, and an inflation risk premium ϕ_π of zero. The suboptimal duration corresponds to the observed average duration of total TDF assets. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. The benchmark parameter values are given in Section 3.4.

solution is equal to the unconstrained solution as described in Section 5.1. Below this age, the portfolio strategy can be roughly described by the following rule of thumb: scarce available financial capital is first used for stock investments, up to the unconstrained optimal level, and then it is used for bond investments. This is only approximately true though. Indeed, at the age of 60, it is clearly visible that the optimal allocation to stocks kinks, which shows that the constraint also reduces the stock exposure at this point. Figure 13(b) compares the median duration of the financial wealth portfolio with the average duration of TDF investment portfolios. We observe that portfolio restrictions may explain the low observed duration early in life, but fails to explain the observed duration later in life.

Figure 14 shows the joint impact of unhedgeable inflation risk and portfolio restrictions on the optimal and observed duration for a relative risk aversion coefficient of 5 and a relative risk aversion coefficient of 2. We observe that our model is able to explain the



(a) Portfolio Shares (b) Comparison of Durations

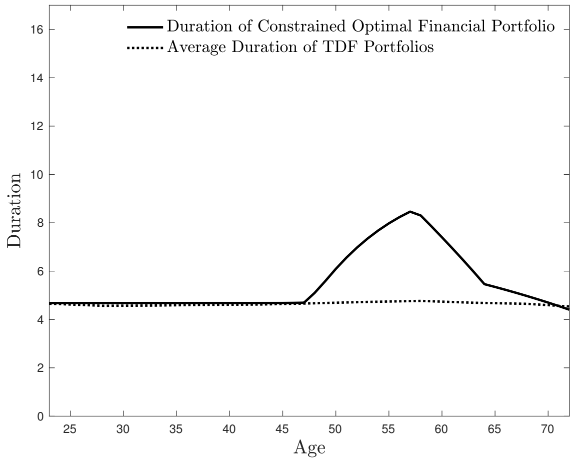
Figure 13. Impact of portfolio restrictions on portfolio strategy and duration. Panel (a) illustrates the median shares of financial wealth invested in a 30-year bond, a risky stock and cash as a function of age. Panel (b) shows the median duration of the optimal financial wealth portfolio (solid line) and the average duration of TDF investment portfolios (dash-dotted line). The figure assumes that the state pension equals 40% of labor income. The benchmark parameter values are given in Section 3.4.

duration puzzle if we set the relative risk aversion coefficient equal to 2 and the inflation rate volatility to 1%. However, one may question whether such parameter values are realistic.

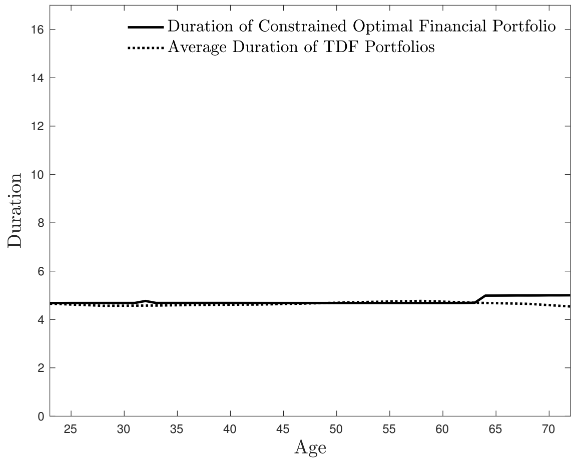
Finally, Figure 15 illustrates the welfare costs associated with implementing a suboptimal duration. The optimal investment strategy takes risk-free labor income into account. Furthermore, the financial market does not include inflation-linked bonds and the optimal portfolio weights have to lie between zero and one. At young ages, the suboptimal duration is assumed to be equal to the optimal duration, because long-term bonds are absent in the optimal investment portfolio. At higher ages, the suboptimal duration corresponds to the observed average duration of total TDF assets. We conclude that welfare losses for middle-aged individuals remain substantial.

6 Non-Time-Separability and Housing

This section briefly indicates the implications of non-time-separable preferences and owner-occupied housing and mortgage wealth for the optimal portfolio allocation over the investor's life-cycle.



(a) $\gamma = 5$



(b) $\gamma = 2$

Figure 14. Joint impact of unhedgeable inflation risk and portfolio restrictions on the optimal and observed duration. This figure illustrates the joint impact of unhedgeable inflation risk and portfolio restrictions on the median duration of the optimal financial wealth portfolio as well as the average duration of TDF investment portfolios. Panel (a) considers a relative risk aversion coefficient of 5, while Panel (b) assumes an relative risk aversion coefficient of 2. The figure assumes that the state pension equals 40% of labor income. Furthermore, we assume no correlation between the Brownian increments, a half-time of the inflation rate of 10 years, an inflation rate volatility of 1%, and an inflation risk premium ϕ_π of zero. The benchmark parameter values are given in Section 3.4.

6.1 Non-Time-Separable Preferences

In the economics and finance literature, it is standard to assume time-separable preferences. However, the literature has also developed various non-time-separable preference specifications. One such common specification is internal habit formation. This section explores the impact of internal habit formation on the optimal portfolio allocation over the investor's life-cycle. More specifically, we assume that the individual behaves in accordance with the ratio internal habit model; see, e.g., Abel (1990), Carroll (2000), Fuhrer (2000), and Gomes and Michaelides (2003). In this model, the investor derives utility from the ratio between current consumption and a habit level which depends on own past consumption levels.

Bilsen, Bovenberg, and Laeven (2019) analytically solve the ratio internal habit model in a setting with stock market risk and interest rate risk. They show that the size of the duration of the optimal investment portfolio is driven by two factors. First, a persistent decrease in the interest rate is less harmful for older individuals than it is for younger individuals. This factor, which is familiar from, e.g., Brennan and Xia (2002) and Merton (2014), causes the duration of the optimal investment portfolio to decrease

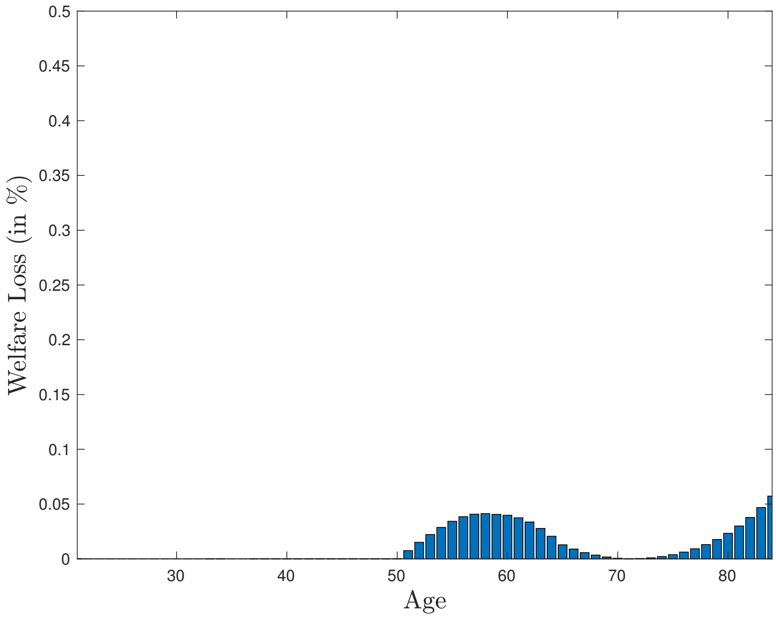


Figure 15. Welfare costs. This figure shows what the welfare loss will be if an investor of a particular age holds an investment portfolio with a suboptimal duration during the first year. After the first year, she holds the optimal investment portfolio for the rest of her life. The optimal investment strategy takes risk-free labor income into account and assumes that the state pension equals 40% of labor income. The financial market does not include inflation-linked bonds and the optimal portfolio weights have to lie between zero and one. Furthermore, we assume no correlation between the Brownian increments, a half-time of the inflation rate of 10 years, an inflation rate volatility of 1%, and an inflation risk premium ϕ_π of zero. At young ages, the suboptimal duration is assumed to be equal to the optimal duration, because long-term bonds are absent in the optimal investment portfolio. At higher ages, the suboptimal duration corresponds to the observed average duration of total TDF assets. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. The benchmark parameter values are given in Section 3.4.

with age. Second, an individual with habit-forming preferences has a lower willingness to substitute consumption over time when old than when young. Intuitively, as the individual becomes older, the duration of remaining lifetime consumption shrinks, and hence the current habit level determines to a larger degree future consumption choices. This factor causes the duration of the optimal investment portfolio to increase with age. The net effect of these two forces leads to a hump-shaped pattern for the duration of the optimal investment portfolio over the investor's life-cycle. As a result, under internal multiplicative habits, it still remains difficult to explain why middle-aged individuals hold an investment portfolio with a low duration.

6.2 Owner-Occupied Housing and Mortgage Wealth

An important component of household wealth not included in the analysis so far is owner-occupied housing and mortgage wealth. For many individuals the house and mortgage are a significant share of total wealth. The house is a long-lived asset that provides an important share of future consumption. House prices are related to interest rates and mortgages also provide interest rate exposure. Therefore, the inclusion of housing and mortgage wealth in the analysis will have an impact on the optimal exposure of an individual's financial wealth to interest rate risk. There are several papers[16] that study the role of the house in the optimal portfolio decision, yet the interaction between housing and interest rate risk has received not much attention. Some notable exceptions are Campbell and Cocco (2003) and Hemert (2010). Campbell and Cocco (2003) study the impact of interest rate risk and inflation risk on the optimal choice of a mortgage contract (i.e., fixed rate versus adjustable rate). They assume however that the choice of the house itself is fixed and does not play a direct role in the optimal portfolio. Hemert (2010) studies a full portfolio choice model with interest rate risk and an endogenous choice of the house. His analysis suggests that housing may justify a low duration of financial wealth. However, Hemert (2010) models the house price such that it does not feature any discount rate variation (very much like the simple stock process we saw in our benchmark setup). It may be insightful to consider a somewhat more general model of the house price, where the house is seen as a claim on future housing services. This will increase the interaction with interest rate movements and may significantly change the optimal asset allocation of the financial wealth portfolio.

7 Conclusion

The problem of optimal portfolio choice over the life-cycle has intrigued many authors at least since Merton (1971). As pointed out by, e.g., Brennan and Xia (2002), Campbell and Viceira (2001) and Merton (2014), a long-term investor should invest a substantial part of her wealth in a long-term bond early in life and reduce bond investments when she gets older. In this paper, we document that the duration of TDF investment portfolios is *not* consistent with this theoretical finding. We have called this finding the duration puzzle. We have explored several extensions of classical portfolio theory to see if our

[16] See, e.g., Sinai and Souleles (2005), Cocco (2005), Yao and Zhang (2005), and Corradin, Fillat, and Vergara-Alert (2014).

rational models can solve the duration puzzle. We have considered the impact of human capital, inflation risk and portfolio restrictions. We conclude that it is difficult to explain the duration puzzle, especially for individuals aged between 35 and 65.

Only when we assume a rather tight portfolio restriction, unhedgeable inflation risk and a relatively low coefficient of relative risk aversion, we were able to produce an optimal life-cycle pattern of the duration that was somewhat close to the observed empirical pattern. A drawback of this explanation is that empirical data suggests a larger relative risk aversion coefficient. Additionally, our portfolio restriction implies that TDF managers cannot take any leverage at all (i.e., no derivatives), which is probably a too restrictive assumption.

The observed life-cycle patterns suggest that TDF managers do not solve the dynamic optimization problem of an 'average' fund participant, but instead reduce the one-period portfolio variance as the target date approaches. They thus ignore the 'Mertonian' hedging demands. It is probably no coincidence that Dimensional (which is advised by Robert Merton) is the only TDF provider that features a notable hedging pattern. The fact that all other TDF managers ignore the 'Mertonian' hedging demands suggests that there is a structural issue. Competitive forces do not seem to lead to implementation of theoretically optimal dynamic strategies. One explanation may be that it is hard for TDFs to market the optimal dynamic strategy. Furthermore, the higher duration of the optimal bond portfolio leads to a higher year-to-year portfolio volatility. If funds are compared by their year-to-year Sharpe ratios, the optimal strategy may appear suboptimal to the average individual saving for retirement.

A Mathematical Proofs

A.1 Derivation of Bond Price Dynamics

This appendix derives the dynamics of the price of a nominal bond. We assume that the economy consists of three state variables: the real interest rate $r(t)$ (with dynamics (3.2)), the dividend payment $D(t)$ (with dynamics (3.3)), and the inflation rate (with dynamics (5.5)). The dynamics of the price of an inflation-linked bond emerges as a special case by setting $\pi(0) = \sigma_\pi = 0$. We can obtain the bond price $p(t, h)$ by computing the following conditional expectation:

$$\begin{aligned} p(t,h) &= \mathbb{E}_t\left[\frac{M(t+h)}{M(t)}\right] \\ &= \mathbb{E}_t\left[\exp\left\{-\int_0^h \left(r(t+v) + \pi(t+v) + \frac{1}{2}\phi^\top \rho\phi\right)\mathrm{d}v + \phi^\top \int_0^h \mathrm{d}Z(t+v)\right\}\right]. \end{aligned} \tag{A1}$$

Here, $\mathbb{E}_t$ denotes the expectation conditional upon the information available at time t.

Equation (A1) shows that the aggregate real interest rate $\bar{r}(t,h) = \int_0^h r(t+v)\mathrm{d}v$ and the aggregate inflation rate $\bar{\pi}(t,h) = \int_0^h \pi(t+v)\mathrm{d}v$ play a key role in determining the nominal bond price. We find that the aggregate real interest rate $\bar{r}(t,h)$ is given by

$$\begin{aligned} \bar{r}(t,h) &= \int_0^h r(t+v)\mathrm{d}v \\ &= \int_0^h \left(e^{-\kappa_r v} r(t) + \left(1 - e^{-\kappa_r v}\right)\bar{r}\right)\mathrm{d}v + \sigma_r \int_0^h \int_0^v e^{-\kappa_r (v-u)}\mathrm{d}Z_r(t+u)\mathrm{d}v \\ &= \int_0^h \left(r(t) + \left(1 - e^{-\kappa_r v}\right)(\bar{r} - r(t))\right)\mathrm{d}v + \sigma_r \int_0^h \int_v^h e^{-\kappa_r (h-u)}\mathrm{d}u\mathrm{d}Z_r(t+v) \\ &= \int_0^h \left(r(t) + \kappa_r B_r(v)(\bar{r} - r(t))\right)\mathrm{d}v + \sigma_r \int_0^h \frac{1}{\kappa_r}\left(1 - e^{-\kappa_r (h-v)}\right)\mathrm{d}Z_r(t+v) \\ &= \int_0^h \mathbb{E}_t\left[r(t+v)\right]\mathrm{d}v + \sigma_r \int_0^h B_r(h-v)\mathrm{d}Z_r(t+v). \end{aligned} \tag{A2}$$

The second equality in (A2) follows from the fact that

$$\begin{aligned} r(t+v) &= e^{-\kappa_r v} r(t) + \left(1 - e^{-\kappa_r v}\right)\bar{r} + \sigma_r \int_0^v e^{-\kappa_r (v-u)}\mathrm{d}Z_r(t+u) \\ &= \mathbb{E}_t\left[r(t+v)\right] + \sigma_r \int_0^v e^{-\kappa_r (v-u)}\mathrm{d}Z_r(t+u). \end{aligned} \tag{A3}$$

We can derive (A3) by repeated substitution. In a similar fashion, we find that the aggregate inflation rate $\bar{\pi}(t,h)$ is given by

$$\bar{\pi}(t,h) = \int_0^h \pi(t+v)\mathrm{d}v = \int_0^h \mathbb{E}_t\left[\pi(t+v)\right]\mathrm{d}v + \sigma_\pi \int_0^h B_\pi(h-v)\mathrm{d}Z_\pi(t+v). \quad \text{(A4)}$$

Substituting (A2) and (A4) into (A1) to eliminate $\int_0^h r(t+v)\mathrm{d}v$ and $\int_0^h \pi(t+v)\mathrm{d}v$, we arrive at

$$\begin{aligned}
p(t,h) &= \exp\left\{-\int_0^h \left(\mathbb{E}_t\left[r(t+v)+\pi(t+v)\right] + \frac{1}{2}\phi^\top \rho \phi\right)\mathrm{d}v\right\} \\
&\quad \mathbb{E}_t\left[\exp\left\{\int_0^h \left(\phi_r - B_r(h-v)\sigma_r\right)\mathrm{d}Z_r(t+v) + \int_0^h \phi_D \mathrm{d}Z_D(t+v)\right.\right. \\
&\qquad \left.\left. + \int_0^h \left(\phi_\pi - B_\pi(h-v)\sigma_\pi\right)\mathrm{d}Z_\pi(t+v)\right\}\right] \\
&= \exp\left\{-\int_0^h \left(\mathbb{E}_t\left[r(t+v)+\pi(t+v)\right] - \lambda_r\sigma_r B_r(v) - \lambda_\pi\sigma_\pi B_\pi(v)\right.\right. \\
&\qquad \left.\left. -\frac{1}{2}B_r^2(v)\sigma_r^2 - \frac{1}{2}B_\pi^2(v)\sigma_\pi^2 - \rho_{r\pi}B_r(v)B_\pi(v)\sigma_r\sigma_\pi\right)\mathrm{d}v\right\} \\
&= \exp\left\{-\int_0^h R(t,v)\mathrm{d}v\right\}.
\end{aligned} \quad \text{(A5)}$$

Here, the instantaneous nominal forward interest rate at adult age t for horizon v, i.e., $R(t,v)$, is defined as follows:

$$\begin{aligned}
R(t,v) &= \mathbb{E}_t\left[r(t+v)+\pi(t+v)\right] - \lambda_r\sigma_r B_r(v) - \lambda_\pi\sigma_\pi B_\pi(v) \\
&\quad - \frac{1}{2}B_r^2(v)\sigma_r^2 - \frac{1}{2}B_\pi^2(v)\sigma_\pi^2 - \rho_{r\pi}B_r(v)B_\pi(v)\sigma_r\sigma_\pi.
\end{aligned} \quad \text{(A6)}$$

The log bond price is given by (this follows from (A5) and (A6))

$$\begin{aligned}
\log p(t,h) = -\int_0^h \Big(& r(t) + \kappa B_r(v)\left(\bar{r} - r(t)\right) + \pi(t) + \kappa_\pi B_\pi(v)\left(\bar{\pi} - \pi(t)\right) - \lambda_r\sigma_r B_r(v) \\
& - \lambda_\pi\sigma_\pi B_\pi(v) - \frac{1}{2}B_r^2(v)\sigma_r^2 - \frac{1}{2}B_\pi^2(v)\sigma_\pi^2 - \rho_{r\pi}B_r(v)B_\pi(v)\sigma_r\sigma_\pi\Big)\mathrm{d}v.
\end{aligned} \quad \text{(A7)}$$

Solving the integral (A7), we arrive at[17]

$$\begin{aligned}
\log p(t,h) &= -r(t)h - (\bar{r} - r(t))\,(h - B_r(h)) - \pi(t)h - (\bar{\pi} - \pi(t))\,(h - B_\pi(h)) \\
&\quad + \frac{\lambda_r \sigma_r}{\kappa_r}\,(h - B_r(h)) + \frac{\lambda_\pi \sigma_\pi}{\kappa_\pi}\,(h - B_\pi(h)) \\
&\quad + \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(h - 2B_r(h) + \frac{1}{2}B_r(2h)\right) + \frac{1}{2}\frac{\sigma_\pi^2}{\kappa_\pi^2}\left(h - 2B_\pi(h) + \frac{1}{2}B_\pi(2h)\right) \\
&\quad + \frac{\rho_{r\pi}\sigma_r\sigma_\pi}{\kappa_r\kappa_\pi}\left(h - B_r(h) - B_\pi(h) + \frac{1}{\kappa_r + \kappa_\pi}\left(1 - e^{-(\kappa_r+\kappa_\pi)h}\right)\right) \\
&= -r(t)B_r(h) - \pi(t)B_\pi(h) - m(h).
\end{aligned} \tag{A8}$$

Here, $m(h)$ is defined as follows:

$$\begin{aligned}
m(h) &= \left(\bar{r} - \frac{\lambda_r\sigma_r}{\kappa_r} - \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\right)(h - B_r(h)) + \frac{1}{4\kappa_r}B_r^2(h)\sigma_r^2 \\
&\quad + \left(\bar{\pi} - \frac{\lambda_\pi\sigma_\pi}{\kappa_\pi} - \frac{1}{2}\frac{\sigma_\pi^2}{\kappa_\pi^2}\right)(h - B_\pi(h)) + \frac{1}{4\kappa_\pi}B_\pi^2(h)\sigma_\pi^2 \\
&\quad + \frac{\rho_{r\pi}\sigma_r\sigma_\pi}{\kappa_r\kappa_\pi}\left(h - B_r(h) - B_\pi(h) + \frac{1}{\kappa_r + \kappa_\pi}\left(1 - e^{-(\kappa_r+\kappa_\pi)h}\right)\right).
\end{aligned} \tag{A9}$$

To calculate how the price of a nominal bond with a fixed maturity date $t+h$ develops as time proceeds (i.e., $t+h$ is fixed but t changes), we apply Itô's lemma to

$$p(t,h) = \exp\left\{-r(t)B_r(h) - \pi(t)B_\pi(h) - m(h)\right\}. \tag{A10}$$

We find

$$\begin{aligned}
\frac{\mathrm{d}p(t,h)}{p(t,h)} &= \Big(R(t,h) - \kappa_r B_r(h)\,(\bar{r} - r(t)) - \kappa_\pi B_\pi(h)\,(\bar{\pi} - \pi(t)) \\
&\qquad + \frac{1}{2}B_r^2(h)\sigma_r^2 + \frac{1}{2}B_\pi^2(h)\sigma_\pi^2 + \rho_{r\pi}B_r(h)B_\pi(h)\sigma_r\sigma_\pi\Big)\mathrm{d}t \\
&\quad - B_r(h)\sigma_r \mathrm{d}Z_r(t) - B_\pi(h)\sigma_\pi \mathrm{d}Z_\pi(t) \\
&= (r(t) + \pi(t) - \lambda_r\sigma_r B_r(h) - \lambda_\pi\sigma_\pi B_\pi(h))\,\mathrm{d}t \\
&\quad - B_r(h)\sigma_r \mathrm{d}Z_r(t) - B_\pi(h)\sigma_\pi \mathrm{d}Z_\pi(t).
\end{aligned} \tag{A11}$$

[17] The first equality follows from $B_r^2(v) = \left(1 - 2e^{-\kappa_r v} + e^{-2\kappa_r v}\right)/\kappa_r^2$ and the second equality follows from $B_r^2(h) = (2B_r(h) - B_r(2h))/\kappa_r$.

A.2 Derivation of Stock Price Dynamics

This appendix starts by deriving the dynamics of the ex-dividend stock price $\widetilde{S}(t)$. We assume that a stock is a claim on future dividends. Hence,

$$\widetilde{S}(t) = \mathbb{E}_t \left[\int_t^\infty \frac{M(s)}{M(t)} D(s) \, \mathrm{d}s \right] \tag{A12}$$

with $D(s)$ the dividend payment at time s (its dynamics are given by (3.3)).

The ex-dividend stock price becomes:

$$\begin{aligned}
\widetilde{S}(t) &= D(t)\mathbb{E}_t \left[\int_t^\infty \frac{M(s)}{M(t)} \frac{D(s)}{D(t)} \, \mathrm{d}s \right] \\
&= D(t) \int_t^\infty \exp\left\{ - \int_t^s \left(r(t) + \kappa_r B_r(v) \left(\bar{r} - r(t)\right) + \frac{1}{2}\phi^\top \rho \phi - \mu_D + \frac{1}{2}\sigma_D^2 \right) \mathrm{d}v \right. \\
&\left. + \phi^\top \int_t^s \mathrm{d}Z(v) - \sigma_r \int_t^s B_r(s-v) \, \mathrm{d}Z_r(v) + \sigma_D \int_t^s \mathrm{d}Z_D(v) \right\} \mathrm{d}s \\
&= D(t) \int_t^\infty \exp\left\{ A_S(s-t) - B_r(s-t) r(t) \right\} \mathrm{d}s,
\end{aligned} \tag{A13}$$

where

$$\begin{aligned}
A_S(h) &= -\Big(h - B_r(h)\Big)\bar{r} + \mu_D h - \lambda_D \sigma_D h + \frac{\sigma_r}{\kappa_r}\Big(h - B_r(h)\Big)(\lambda_r - \sigma_D \rho_{rD}) \\
&+ \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(h - 2B_r(h) + \frac{1}{2}B_r(2h) \right).
\end{aligned} \tag{A14}$$

Now the stock price process can be written as follows:

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{S}(t)}{\widetilde{S}(t)} &= \frac{\mathrm{d}D(t)}{D(t)} + \frac{\int_t^\infty \widetilde{S}(t,s)\left(\frac{\partial A_S(s-t)}{\partial t} - \frac{\partial B_r(s-t)}{\partial t} r(t) \right) \mathrm{d}s - \widetilde{S}(t,t)}{\widetilde{S}(t)} \mathrm{d}t \\
&- \frac{\int_t^\infty \widetilde{S}(t,s) B_r(s-t) \, \mathrm{d}s}{\widetilde{S}(t)} \mathrm{d}r(t) - \frac{\mathrm{d}D(t)}{D(t)} \frac{\int_t^\infty \widetilde{S}(t,s) B_r(s-t) \, \mathrm{d}s}{\widetilde{S}(t)} \mathrm{d}r(t) \\
&+ \frac{1}{2} \frac{\int_t^\infty \widetilde{S}(t,s) B_r^2(s-t) \, \mathrm{d}s}{\widetilde{S}(t)} (\mathrm{d}r(t))^2,
\end{aligned} \tag{A15}$$

where

$$\widetilde{S}(t,s) = D(t) \exp\left\{ A_S(s-t) - B_r(s-t) r(t) \right\}. \tag{A16}$$

This simplifies to:

$$\frac{\mathrm{d}\widetilde{S}(t)}{\widetilde{S}(t)} = -\frac{D(t)}{\widetilde{S}(t)}\,\mathrm{d}t + \Big(r(t) - \lambda_r D_S(t)\sigma_r + \lambda_D\sigma_D\Big)\,\mathrm{d}t - D_S(t)\sigma_r\,\mathrm{d}Z_r(t) + \sigma_D\,\mathrm{d}Z_D(t) \tag{A17}$$

with

$$D_S(t) = \frac{\int_t^\infty \widetilde{S}(t,s)B_r(s-t)\,\mathrm{d}s}{\widetilde{S}(t)}. \tag{A18}$$

Adding the dividend yield to the price process (A17), we arrive at the equation in the main text (3.8)

A.3 Derivation of Optimal Portfolio Weights

This appendix derives the optimal portfolio weights. We assume that the economy consists of three state variables: the real interest rate $r(t)$ (with dynamics (3.2)), the dividend payment $D(t)$ (with dynamics (3.3)), and the inflation rate (with dynamics (5.5)). We assume that the investor has the opportunity to invest in three risky assets: an inflation-linked bond with fixed time to maturity h, a risky stock, and a nominal bond with fixed time to maturity h_N. The dynamics of the bond prices and the stock price are derived in Appendices A.1 and A.2, respectively.

We start by deriving the optimal (real) consumption choice $c^*(t)$. Denote by $\mathcal{L}$ the Lagrangian which is given by

$$\begin{aligned}\mathcal{L} &= \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}c(t)^{1-\gamma}\,\mathrm{d}t\right] + y\left(W(0) - \mathbb{E}\left[\int_0^T M(t)c(t)\,\mathrm{d}t\right]\right)\\ &= \int_0^T \mathbb{E}\left[e^{-\delta t}\frac{1}{1-\gamma}c(t)^{1-\gamma} - yM(t)c(t)\right]\mathrm{d}t + yW(0).\end{aligned} \tag{A19}$$

Here $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. The individual aims to maximize $e^{-\delta t}\frac{1}{1-\gamma}c(t)^{1-\gamma} - yM(t)c(t)$. The optimal consumption choice $c^*(t)$ satisfies the following first-order optimality condition:

$$e^{-\delta t}\left(c^*(t)\right)^{-\gamma} = yM(t). \tag{A20}$$

After solving the first-order optimality condition, we obtain the following optimal consumption choice:

$$c^*(t) = \left(e^{\delta t} y M(t)\right)^{-\frac{1}{\gamma}}. \tag{A21}$$

A verification that the optimal solution to the Lagrangian equals the optimal solution to the investor's maximization problem (3.11) (see, e.g., Karatzas and Shreve (1998)) completes the proof.

Substituting the expression for the stochastic discount factor $M(t) = \exp\left\{-\int_0^t \left(r(t) + \frac{1}{2}\phi^\top \rho \phi\right) \mathrm{d}t + \phi^\top \int_0^t \mathrm{d}Z(t)\right\}$ into (A21), we find

$$c^*(t) = c^*(0) \exp\left\{\frac{1}{\gamma} \int_0^t \left(r(s) + \frac{1}{2}\phi^\top \rho \phi - \delta\right) \mathrm{d}s - \frac{1}{\gamma}\phi^\top \int_0^t \mathrm{d}Z(s)\right\}. \tag{A22}$$

Denote by $V^*(t)$ the market-consistent value at adult age t of current and future optimal consumption choices. We define $V^*(t)$ as follows:

$$\begin{aligned} V^*(t) &= \int_0^{T-t} \mathbb{E}_t \left[\frac{M(t+h)}{M(t)} c^*(t+h)\right] \mathrm{d}h \\ &= c^*(t) \int_0^{T-t} \mathbb{E}_t \left[\frac{M(t+h)}{M(t)} \frac{c^*(t+h)}{c^*(t)}\right] \mathrm{d}h = c^*(t) A^*(t), \end{aligned} \tag{A23}$$

where $A^*(t)$ denotes the optimal annuity factor at adult age t:

$$A^*(t) = \int_0^{T-t} \mathbb{E}_t \left[\frac{M(t+h)}{M(t)} \frac{c^*(t+h)}{c^*(t)}\right] \mathrm{d}h = \int_0^{T-t} \exp\left\{-d^*(t,h)h\right\} \mathrm{d}h. \tag{A24}$$

Here, $d^*(t,h)$ represents the market-consistent discount rate at adult age t for horizon

$h \geq 0$. Straightforward computations show that

$$
\begin{aligned}
d^*(t,h) = \frac{1}{h}\Bigg[&\left(1-\frac{1}{\gamma}\right)\int_0^h \left(r(t)+\kappa_r B_r(v)\left(\bar{r}-r(t)\right)+\frac{1}{2}\phi^\top \rho \phi\right)\mathrm{d}v \\
&-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\int_0^h \left(\phi_r - B_r(v)\sigma_r\right)^2\mathrm{d}v \\
&-\left(1-\frac{1}{\gamma}\right)^2\rho_{rD}\int_0^h \left(\phi_r - B_r(v)\sigma_r\right)\phi_D\,\mathrm{d}v \\
&-\left(1-\frac{1}{\gamma}\right)^2\rho_{r\pi}\int_0^h \left(\phi_r - B_r(v)\sigma_r\right)\phi_\pi\,\mathrm{d}v\Bigg] \\
-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2&\phi_D^2-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\phi_\pi^2-\left(1-\frac{1}{\gamma}\right)^2\rho_{D\pi}\phi_D\phi_\pi.
\end{aligned}
\tag{A25}
$$

The quantity $\log V^*(t)$ evolves according to (this follows from (A21), (A24) and (A25))

$$
\begin{aligned}
\mathrm{d}\log V^*(t) &= \mathrm{d}\log c^*(t)+\mathrm{d}\log A^*(t) \\
&= (\ldots)\,\mathrm{d}t-\left(\frac{1}{\gamma}\phi_r + D_A(t)\sigma_r\right)\mathrm{d}Z_r(t)-\frac{1}{\gamma}\phi_D\,\mathrm{d}Z_D(t)-\frac{1}{\gamma}\phi_\pi\,\mathrm{d}Z_\pi(t).
\end{aligned}
\tag{A26}
$$

Here, $D_A(t)$ represents the duration of the optimal annuity factor:

$$
D_A(t) = \left(1-\frac{1}{\gamma}\right)\int_0^{T-t}\frac{V^*(t,h)}{V^*(t)}B_r(h)\,\mathrm{d}h, \tag{A27}
$$

where $V^*(t,h) = c^*(t)\exp\{-d^*(t,h)h\}$.

Log total wealth evolves according to:

$$
\begin{aligned}
\mathrm{d}\log W(t) = (\ldots)\,\mathrm{d}t &- \left[\omega_P(t)B_r\left(h\right)+\omega_S(t)D_S(t)+\omega_p(t)B_r\left(h_N\right)\right]\sigma_r\mathrm{d}Z_r(t) \\
&+\omega_S(t)\sigma_D\mathrm{d}Z_D(t)-\omega_p(t)B_\pi\left(h_N\right)\sigma_\pi\mathrm{d}Z_\pi(t).
\end{aligned}
\tag{A28}
$$

Comparing (A28) with (A26), we find

$$
\omega_P^*(t) = \frac{1}{\gamma}\frac{\phi_r}{B_r\left(h\right)\sigma_r}+\frac{D_A(t)}{B_r\left(h\right)}-\omega_S^*(t)\frac{D_S(t)}{B_r(h)}-\omega_p^*(t)\frac{B_r(h_N)}{B_r(h)}, \tag{A29}
$$

$$
\omega_S^*(t) = -\frac{1}{\gamma}\frac{\phi_D}{\sigma_D}, \tag{A30}
$$

$$
\omega_p^*(t) = \frac{1}{\gamma}\frac{\phi_\pi}{B_\pi\left(h_N\right)\sigma_\pi}. \tag{A31}
$$

Note that the portfolio strategies presented in Sections 4.2 and 5.2.1 arise as a special case.

A.4 Impact of Human Capital on Optimal Portfolio Weights

This appendix explores the impact of human capital on the optimal portfolio weights. We consider an economy with two state variables: the interest rate $r(t)$ (with dynamics (3.2)), and the stock price $S(t)$ (with dynamics (3.8)). Denote by $O(t)$ and $H(t)$ outside income and human capital at adult age t, respectively. We define human capital as follows:

$$H(t) \equiv \int_0^{T-t} H(t,h)\,\mathrm{d}h, \tag{A32}$$

where

$$H(t,h) = \mathbb{E}_t\left[\frac{M(t+h)}{M(t)}O(t+h)\right] \tag{A33}$$

with

$$O(t+h) = \begin{cases} 1 & \text{if } t+h < T_R \\ s & \text{if } t+h \geq T_R. \end{cases} \tag{A34}$$

Here, T_R denotes the age at which the individual retires and s represents the social security payment.

Straightforward computations show

$$\mathrm{d}H(t) = \left(r(t) - \lambda_r\sigma_r D_H(t)\right) H(t)\,\mathrm{d}t - D_H(t)\sigma_r H(t)\,\mathrm{d}Z_r(t) - O(t)\,\mathrm{d}t, \tag{A35}$$

where

$$D_H(t) = \int_0^{T-t} \frac{H(t,h)}{H(t)} B_r(h)\,\mathrm{d}h \tag{A36}$$

denotes the duration of human capital.

Financial wealth $F(t)$ evolves as follows:

$$\frac{\mathrm{d}F(t)}{F(t)} = (\ldots)\,\mathrm{d}t - \left[\widehat{\omega}_P(t)B_r(h) + \widehat{\omega}_S(t)D_S(t)\right]\sigma_r\mathrm{d}Z_r(t) + \widehat{\omega}_S(t)\sigma_D\mathrm{d}Z_D(t). \tag{A37}$$

Hence, total wealth $W(t) = H(t) + F(t)$ satisfies

$$\begin{aligned}
\mathrm{d}W(t) &= \mathrm{d}H(t) + \mathrm{d}F(t) \\
&= (\ldots)\,\mathrm{d}t + \widehat{\omega}_S(t)\sigma_D \frac{F(t)}{W(t)} W(t)\mathrm{d}Z_D(t) \\
&\quad - \left[\widehat{\omega}_P(t)B_r\left(h\right)\frac{F(t)}{W(t)} + \widehat{\omega}_S(t)D_S(t)\frac{F(t)}{W(t)} + D_H(t)\frac{H(t)}{W(t)}\right]\sigma_r W(t)\mathrm{d}Z_r(t).
\end{aligned} \tag{A38}$$

Comparing (3.10) with (A38), we arrive at (5.2) and (5.3).

A.5 Impact of Incomplete Asset Menu on Optimal Portfolio Weights

This appendix derives the optimal portfolio weights for a terminal wealth investor without outside income in case the market for inflation risk hedging is incomplete. We assume three state variables: the real interest rate $r(t)$ (with dynamics (3.2)), the stock price $S(t)$ (with dynamics (3.8)), and the inflation rate (with dynamics (5.5)). In addition, we assume that the duration of the stock is zero (i.e., $D_S(t) = 0$ for all t). We will first solve the problem for a general asset menu, which may or may not be complete. Subsequently, we will specify the asset menu and compare the solution for a complete asset menu with the solution for an incomplete asset menu.

The investor maximizes utility of terminal wealth $u\left(W(T_R)\right)$ subject to the following wealth dynamics:

$$\mathrm{d}W(t) = \left(r(t) + \omega(t)^{\top}\left[\mu(t) - r(t)\right]\right)W(t)\mathrm{d}t + \omega(t)^{\top}\Sigma W(t)\mathrm{d}Z(t), \tag{A39}$$

where $\omega(t)$ is the vector of portfolio weights, $\mu(t)$ is the vector of mean returns for all available assets and Σ is the matrix with the assets' exposures to the different shocks (each row corresponding to a different asset).

The value function is given by:

$$f\left(W(t), r(t), t\right) = \max_{\omega(t)}\ \mathbb{E}_t\left[\frac{1}{1-\gamma}W(T_R)^{1-\gamma}\right]. \tag{A40}$$

The Hamiltonian-Jacobi-Bellman (HJB) equation is now defined as follows:

$$
\begin{aligned}
0 = \max_{\omega(t)} \bigg\{ & f_W W(t)(r(t) - \omega(t)^\top \Sigma\rho\phi) + \frac{1}{2} f_{WW} W(t)^2 \omega(t)^\top \Sigma\rho\Sigma^\top \omega(t) \\
& + f_{Wr} W(t)\omega(t)^\top \Sigma\rho e_1 \sigma_r + f_r \kappa_r (\bar{r} - r(t)) + \frac{1}{2} f_{rr} \sigma_r^2 + f_t \bigg\}
\end{aligned}
\tag{A41}
$$

with $e_1 = (1, 0, 0)^\top$ and

$$
\rho = \begin{pmatrix} 1 & \rho_{rD} & \rho_{r\pi} \\ \rho_{rD} & 1 & \rho_{D\pi} \\ \rho_{r\pi} & \rho_{D\pi} & 1 \end{pmatrix}. \tag{A42}
$$

The first-order optimality condition is given by

$$
0 = -f_W W(t)\Sigma\rho\phi + f_{WW} W(t)^2 \Sigma\rho\Sigma^\top \omega^*(t) + f_{Wr} W(t)\Sigma\rho e_1 \sigma_r. \tag{A43}
$$

Hence, the optimal solution is

$$
\omega^*(t) = \frac{f_W}{f_{WW} W(t)} \left(\Sigma\rho\Sigma^\top\right)^{-1} \Sigma\rho\phi - \frac{f_{Wr}}{f_{WW} W(t)} \left(\Sigma\rho\Sigma^\top\right)^{-1} \Sigma\rho e_1 \sigma_r. \tag{A44}
$$

Let us conjecture that the value function is given by

$$
f\left(W(t), r(t), t\right) = \frac{W(t)^{1-\gamma}}{1-\gamma} \psi\left(r(t), T_R - t\right) \tag{A45}
$$

where

$$
\psi\left(r(t), T_R - t\right) = \exp\left\{(1-\gamma)\left(a\left(T_R - t\right) + B_r\left(T_R - t\right) r(t)\right)\right\} \tag{A46}
$$

with

$$
\begin{aligned}
a\left(T_R - t\right) = {} & \bar{r}\left[T_R - t - B_r\left(T_R - t\right)\right] + \frac{1}{2\gamma}\left(T_R - t\right)\phi^\top \Omega \phi \\
& - \frac{1-\gamma}{\gamma}\frac{\sigma_r}{\kappa_r}\left[T_R - t - B_r\left(T_R - t\right)\right]\phi^\top \Omega e_1 \\
& + \frac{(1-\gamma)^2}{2\gamma}\frac{\sigma_r^2}{\kappa_r^2}\left[T_R - t - B_r\left(T_R - t\right) - \frac{\kappa_r}{2} B_r^2\left(T_R - t\right)\right]\left(e_1^\top \Omega e_1 - 1\right) \\
& + \frac{1-\gamma}{2}\frac{\sigma_r^2}{\kappa_r^2}\left[T_R - t - B_r\left(T_R - t\right) - \frac{\kappa_r}{2} B_r^2\left(T_R - t\right)\right].
\end{aligned}
\tag{A47}
$$

Here, Ω is defined as follows:

$$\Omega = \rho\Sigma^\top \left(\Sigma\rho\Sigma^\top\right)^{-1} \Sigma\rho. \tag{A48}$$

The derivatives are

$$f_W = (1-\gamma) f\left(W(t), r(t), t\right) W(t)^{-1}, \tag{A49}$$
$$f_{WW} = -\gamma(1-\gamma) f\left(W(t), r(t), t\right) W(t)^{-2}, \tag{A50}$$
$$f_{Wr} = (1-\gamma)^2 f\left(W(t), r(t), t\right) W(t)^{-1} B_r\left(T_R - t\right). \tag{A51}$$

Hence, the optimal solution becomes

$$\omega^*(t) = -\frac{1}{\gamma}\left(\Sigma\rho\Sigma^\top\right)^{-1}\Sigma\rho\phi - \frac{\gamma-1}{\gamma}\left(\Sigma\rho\Sigma^\top\right)^{-1}\Sigma\rho e_1 B_r\left(T_R - t\right)\sigma_r. \tag{A52}$$

It is now straightforward (yet tedious) to verify that the conjectured value function indeed satisfies the HJB equation.

Now let us consider the impact of an incomplete asset menu by comparing two situation. One in which the investor has access to three risky assets: a risky stock, an inflation-linked bond and a nominal bond. Since these three assets (plus the instantaneously risk-free asset) jointly span the three risk dimensions, the asset menu is complete and the investor can obtain any combination of risk she prefers. Second, let us consider the case where the investor has access to only two assets: a risky stock and a nominal bond. In that case, the two assets can *not* span all three risk dimensions and hence the asset menu is *in*complete.

In the three asset case, $\mu(t)$ and Σ are equal to:

$$\mu_C(t) = \begin{pmatrix} r(t) + \lambda_D\sigma_D \\ r(t) - \lambda_r\sigma_r B_r\left(h\right) \\ r(t) - \lambda_r\sigma_r B_r\left(h_N\right) - \lambda_\pi\sigma_\pi B_\pi\left(h_N\right) \end{pmatrix} \text{ and } \Sigma_C = \begin{pmatrix} 0 & \sigma_D & 0 \\ -B_r\left(h\right)\sigma_r & 0 & 0 \\ -B_r\left(h_N\right)\sigma_r & 0 & -B_\pi\left(h_N\right)\sigma_\pi \end{pmatrix}.$$

And in the two asset case, we have:

$$\mu_I(t) = \begin{pmatrix} r(t) + \lambda_D\sigma_D \\ r(t) - \lambda_r\sigma_r B_r\left(h_N\right) - \lambda_\pi\sigma_\pi B_\pi\left(h_N\right) \end{pmatrix} \text{ and } \Sigma_I = \begin{pmatrix} 0 & \sigma_D & 0 \\ -B_r\left(h_N\right)\sigma_r & 0 & -B_\pi\left(h_N\right)\sigma_\pi \end{pmatrix}.$$

If we assume that the Brownian increments are uncorrelated, this leads to the following solution in the two situations:

$$\omega_C^*(t) = \frac{1}{\gamma}\begin{pmatrix} -\frac{\phi_D}{\sigma_D} \\ \frac{\phi_r}{B_r(h)\sigma_r} - \frac{\phi_\pi}{B_\pi(h_N)\sigma_\pi}\frac{B_r(h_N)}{B_r(h)} \\ \frac{\phi_\pi}{B_\pi(h_N)\sigma_\pi} \end{pmatrix} + \frac{\gamma-1}{\gamma}\begin{pmatrix} 0 \\ \frac{B_r(T_R-t)}{B_r(h)} \\ 0 \end{pmatrix}, \tag{A53}$$

$$\omega_I^*(t) = \frac{1}{\gamma}\begin{pmatrix} -\frac{\phi_D}{\sigma_D} \\ \frac{\phi_r B_r(h_N)\sigma_r + \phi_\pi B_\pi(h_N)\sigma_\pi}{B_r^2(h_N)\sigma_r^2 + B_\pi^2(h_N)\sigma_\pi^2} \end{pmatrix} + \frac{\gamma-1}{\gamma}\begin{pmatrix} 0 \\ \frac{B_r(T_R-t)}{B_r(h_N)}\left(1+\frac{B_\pi^2(h_N)\sigma_\pi^2}{B_r^2(h_N)\sigma_r^2}\right)^{-1} \end{pmatrix}. \tag{A54}$$

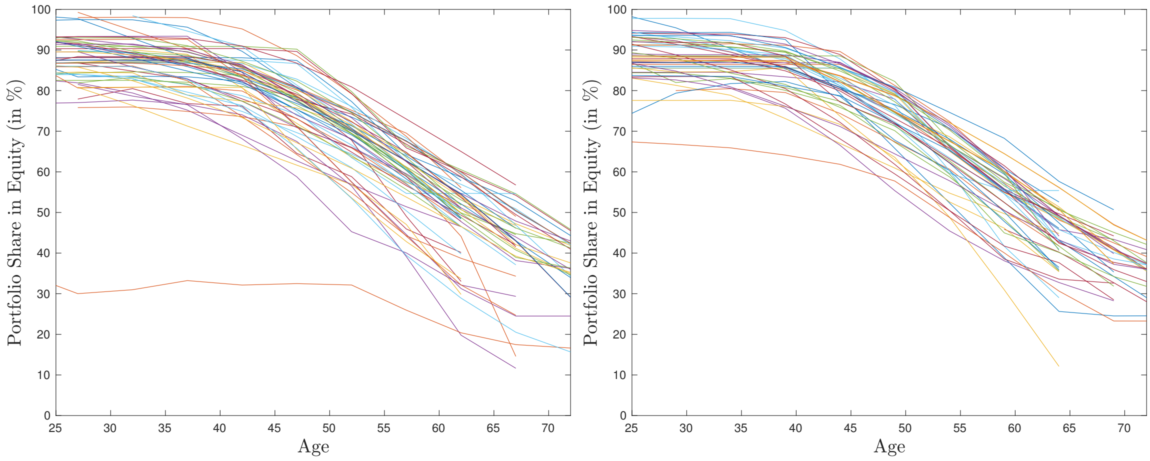
The durations of optimal total wealth are given by

$$D_C(T_R-t) = \frac{1}{\gamma}\frac{\phi_r}{\sigma_r} + \frac{\gamma-1}{\gamma}B_r\left(T_R-t\right), \tag{A55}$$

$$D_I(T_R-t) = D_C(T_R-t)\frac{1}{1+b^2} + \frac{1}{\gamma}\frac{\phi_\pi}{\sigma_r}\frac{b}{1+b^2}, \tag{A56}$$

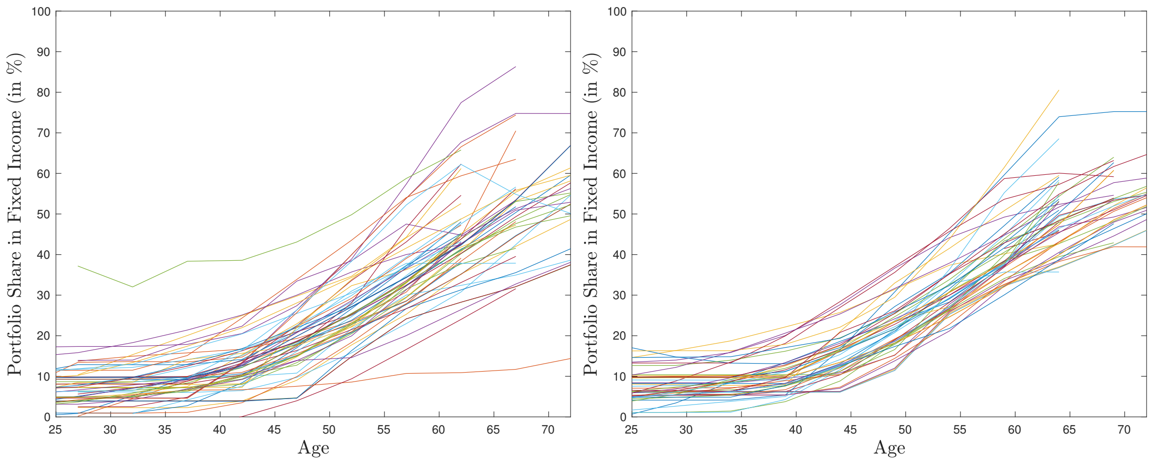
where $b = B_\pi^2\left(h_N\right)\sigma_\pi^2/\left(B_r^2\left(h_N\right)\sigma_r^2\right)$ is the ratio between inflation risk volatility and real interest rate risk volatility. In the incomplete asset menu case, the nominal bond serves a dual purpose: it is used to provide both real interest rate risk exposure and inflation risk exposure. Consider the situation where the reward for taking inflation risk is zero ($\phi_\pi = 0$). In that case, the optimal duration is equal to the optimal duration in the complete market case multiplied by $1/\left(1+b^2\right)$. The bigger the inflation risk volatility, the less useful the long-term bond becomes and the lower the optimal duration will be. On the other hand, a positive inflation risk premium will increase the optimal duration.

B Additional Figures



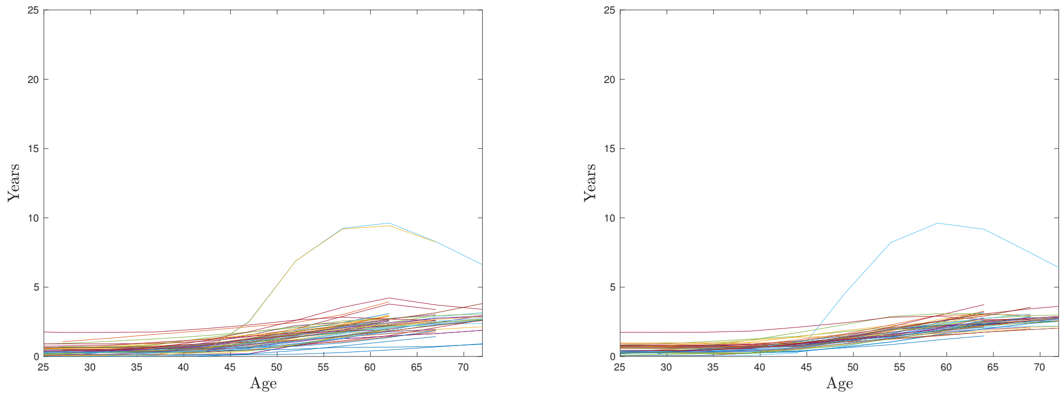
(a) Equity Exposures 2017 (b) Equity Exposures 2019

Figure 16. Variation in TDF equity exposure over the life-cycle. This figure shows for all TDF series, the share of assets invested in equity as a function of age. Panel (a) shows 2017 data and panel (b) 2019 data. The figure assumes that the target date corresponds to an age of 65.



(a) Fixed Income Exposures 2017 (b) Fixed Income Exposures 2019

Figure 17. Variation in TDF fixed income exposure over the life-cycle. This figure shows for all TDF series, the share of assets invested in fixed income as a function of age. Panel (a) shows 2017 data and panel (b) 2019 data. The figure assumes that the target date corresponds to an age of 65.



(a) Contr. of Fixed Income to Duration 2017 (b) Contr. of Fixed Income to Duration 2019

Figure 18. Variation in fixed income contributions over the life-cycle. This figure shows for all TDF series, the contribution (in years) of the fixed income portfolio to the overall portfolio duration as a function of age. Panel (a) shows 2017 data and panel (b) 2019 data. The figure assumes that the target date corresponds to an age of 65.

C Numerical Solution Technique

We determine the optimal consumption and portfolio policies using numerical backward induction. We start by discretizing both the time and the state space. We first specify discrete points in the state space, called grid points, for the final time period. For each grid point, we determine the optimal consumption choice, the optimal portfolio choice and the level of the value function. For the final period these values are trivial, since the individual simply consumes any remaining wealth. We then move one period back in time. We first derive the optimal portfolio decision for all points on the state space grid. Subsequently, we determine the optimal consumption choice using the endogenous grid method proposed by Carroll (2006). This allows us to exploit the analytical first-order condition of the intertemporal consumption problem, which means we do not need to numerically search for the solution. Finding the optimal portfolio weights and optimal consumption policies does require us to evaluate the expected value of the utility function next period. We do so by numerical integration over the state space using Gaussian quadrature. Whenever the integration algorithm requires points that are not on the grid, we use an interpolation technique. In particular, we linearly interpolate a certainty equivalent measure: the certain and flat level of consumption that would deliver the level of utility in the grid point. We then convert the interpolated certainty equivalent back into utility terms. The idea behind this approach is that in the unconstrained problem, this certainty equivalent would be a linear function of the endogenous state (wealth) and hence our procedure would yield the exact utility value for wealth levels not on our grid. To make sure the algorithm never has to evaluate a realization where next period's wealth is negative, we impose the assumption that portfolio shares (as shares of total wealth) are always continuously rebalanced between two discrete periods. We tested the algorithm using the complete market problem without restrictions, for which we have the closed-form solution in continuous time. Using one year time steps the numerical solution is visually indistinguishable from the closed-form solution depicted in Figure 8 in the main text.

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